

# On solutions of ultradiscrete Painlevé II equation with parity variables

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## Abstract

We introduce a simultaneous ultradiscrete Painlevé II equation with parity variables, which is shown to be more suitable for studying two-parameter solutions than the single second-order ultradiscrete Painlevé II equation with parity variables. We investigate several types of two-parameter solutions and the solutions which are related with the ultradiscrete limit of determinant type solutions of  $q$ -Painlevé II.

## 1 Introduction

In the 1990s', discrete analogues of the Painlevé equations had been found by considering alternatives of Painlevé property. Ramani and Grammaticos [9] found the  $q$ -discrete Painlevé II equation, which has several expression. In this paper we adopt the expression of  $q$ -PII as follows:

$$(z(q\tau)z(\tau) + 1)(z(\tau)z(q^{-1}\tau) + 1) = \frac{a\tau^2 z(\tau)}{\tau - z(\tau)}. \quad (1)$$

Hamamoto, Kajiwara and Witte [1] investigated special solutions of  $q$ -PII expressed in terms of determinants whose elements are expressed by  $q$ -Airy function in the case  $a = q^{2N+1}$  and  $N \in \mathbb{Z}$ . Another approach to investigate solutions of  $q$ -Painlevé equations is to investigate solutions in the case  $q = 0$ . The case  $q = 0$  can be realised by the ultradiscrete limit, and it is profitable to introduce a parity variable so that the ultradiscrete limit of special solutions of determinant type is considered (see [5, 6, 7, 3]).

The ultradiscrete Painlevé II equation with parity variables (abbreviated to p-ud PII) is described by using the parity variable  $\zeta_m \in \{+1, -1\}$  and the amplitude variable  $Z_m \in \mathbb{Z}$ . We assume  $Q < 0$  and set  $a = \exp(A/\varepsilon)$ ,  $q = \exp(Q/\varepsilon)$ ,  $\tau = q^m$ ,  $z(q^m) = \zeta_m \exp(Z_m/\varepsilon)$  ( $\zeta_m \in \{\pm 1\}$ ,  $Z_m \in \mathbb{R}$ ) in Eq.(1). By

letting  $\varepsilon \rightarrow +0$ , we obtain p-ud PII

$$\begin{aligned}
& \max \left[ Z_{m+1} + 3Z_m + Z_{m-1} + S(\zeta_{m+1}\zeta_m\zeta_{m-1}), Z_{m+1} + 2Z_m + S(\zeta_{m+1}), \right. \\
& \quad 2Z_m + Z_{m-1} + S(\zeta_{m-1}), Z_m + S(\zeta_m), Z_m + A + 2mQ + S(\zeta_m), \\
& \quad Z_{m+1} + 2Z_m + Z_{m-1} + mQ + S(-\zeta_{m+1}\zeta_{m-1}), \\
& \quad \left. Z_{m+1} + Z_m + mQ + S(-\zeta_{m+1}\zeta_m), Z_m + Z_{m-1} + mQ + S(-\zeta_m\zeta_{m-1}) \right] \\
& = \max \left[ Z_{m+1} + 3Z_m + Z_{m-1} + S(-\zeta_{m+1}\zeta_m\zeta_{m-1}), Z_{m+1} + 2Z_m + S(-\zeta_{m+1}), \right. \\
& \quad 2Z_m + Z_{m-1} + S(-\zeta_{m-1}), Z_m + S(-\zeta_m), Z_m + A + 2mQ + S(-\zeta_m), \\
& \quad Z_{m+1} + 2Z_m + Z_{m-1} + mQ + S(\zeta_{m+1}\zeta_{m-1}), \\
& \quad \left. Z_{m+1} + Z_m + mQ + S(\zeta_{m+1}\zeta_m), Z_m + Z_{m-1} + mQ + S(\zeta_m\zeta_{m-1}), mQ \right],
\end{aligned} \tag{2}$$

where the function  $S : \{+1, -1\} \rightarrow \{0, -\infty\}$  is defined by

$$S(\omega) := \begin{cases} 0 & (\omega = +1), \\ -\infty & (\omega = -1). \end{cases} \tag{3}$$

For details, see [5, 3].

Isojima and the authors [3] studied further the ultradiscrete limit of the determinant-type solutions [1] in the case  $a = q^{2N+1}$  and  $N \in \mathbb{Z}_{\geq 0}$  by following the earlier studies [5, 7]. Consequently, some special solutions of p-ud PII were obtained. An example of solutions to p-ud PII given in [3] is described as

$$(\zeta_m, Z_m) = \begin{cases} (+1, 3m + 21) & (m \leq -19) \\ (+1, 29) & (m = -18) \\ (+1, 33) & (m = -17) \\ (+1, -32) & (m = -16) \\ (-1, 32) & (m = -15) \\ (+1, 30) & (m = -14) \\ (-1, -30) & (m = -13) \\ (+1, 30) & (m = -12) \\ (+1, 16) & (m = -11) \\ (-1, -16) & (m = -10) \\ (+1, -3m) & (-9 \leq m \leq -1) \\ ((-1)^m, 0) & (m \geq 0), \end{cases} \tag{4}$$

where the parameters are chosen as  $Q = -3$  and  $A = 7Q$ .

In this paper, we investigate the p-ud PII (Eq.(2)) and related equations. It has been known that p-ud equation may not have uniqueness of solutions, and we explain it in the case of the forward evolution of Eq.(2). The unique evolution is assigned to the case that  $(\zeta_{m+1}, Z_{m+1})$  is determined uniquely by

Eq.(2) and the values  $(\zeta_{m-1}, Z_{m-1})$  and  $(\zeta_m, Z_m)$ . Conversely, the indefinite evolution is assigned to the case that  $(\zeta_{m+1}, Z_{m+1})$  is not determined uniquely by them. Here we examine the uniqueness and indefiniteness by using the solution in Eq.(4). Let us investigate the forward evolution with the initial values  $(\zeta_{-17}, Z_{-17}) = (+1, 33)$  and  $(\zeta_{-16}, Z_{-16}) = (+1, -32)$ . We substitute  $m = -16$  into Eq.(2), and we have

$$\begin{aligned} & \max[Z_{-15} - 63 + S(\zeta_{-15}), 43, Z_{-15} + 17 + S(-\zeta_{-15})] \\ &= \max[Z_{-15} - 63 + S(-\zeta_{-15}), Z_{-15} + 17 + S(\zeta_{-15}), 49]. \end{aligned} \quad (5)$$

If  $\zeta_{-15} = +1$ , then we have  $\max[Z_{-15} - 63, 43] = \max[Z_{-15} + 17, 49]$  and it turns out that there is no solution. Hence we have  $\zeta_{-15} = -1$ ,  $\max[Z_{-15} + 17, 43] = \max[Z_{-15} - 63, 49]$  and  $Z_{-15} = 32$ . Therefore the evolution to  $(\zeta_{-15}, Z_{-15}) = (-1, 32)$  is unique. Next we substitute  $m = -15$  into Eq.(2). Then we have

$$\begin{aligned} & \max[Z_{-14} + 77 + S(-\zeta_{-14}), Z_{-14} + 77 + S(\zeta_{-14}), 45] \\ &= \max[101, Z_{-14} + 77 + S(\zeta_{-14}), Z_{-14} + 77 + S(-\zeta_{-14})]. \end{aligned} \quad (6)$$

If  $Z_{-14} \geq 24$ , then the ultradiscrete equation is satisfied. Thus the evolution to  $(\zeta_{-14}, Z_{-14})$  is indefinite.

In the general setting, the condition that the forward evolution in Eq.(2) is unique is determined in Proposition 3. Moreover, if the evolution is unique, then the amplitude function  $Z_{m+1}$  is written in a simpler form

$$Z_{m+1} = -Z_m + \max[0, A + 2mQ - \max[0, mQ - Z_m] - \max[0, Z_{m-1} + Z_m]]. \quad (7)$$

However it is shown that Eq.(2) may not govern the solution for all  $m \in \mathbb{Z}$ . Namely, there exists no solution to single ud-PII such that any forward and backward evolution are unique for all  $m \in \mathbb{Z}$  (see Theorem 2).

In order to avoid indefinite evolution, we introduce another variable to p-ud PII. We set  $\tau = q^m$  and  $y(q^{m+1}) = z(q^{m+1})z(q^m) + 1$  in Eq.(1). Then Eq.(1) is written as the simultaneous equation

$$y(q^{m+1})y(q^m) = \frac{aq^{2m}z(q^m)}{q^m - z(q^m)}, \quad y(q^{m+1}) = z(q^{m+1})z(q^m) + 1. \quad (8)$$

Note that the introduction of the variable  $y(q^m)$  is essentially due to Murata [8] for ultradiscrete Painlevé II equation without parity variables. Let  $(\eta_m, Y_m)$  be the parity variable and the amplitude function determined by  $y(q^m) = \eta_m \exp(Y_m/\varepsilon)$ . Then the corresponding p-ud equation is written as

$$\max[mQ - Z_m + Y_{m+1} + Y_m + S(\eta_{m+1}\eta_m), \quad (9)$$

$$\begin{aligned} & Y_{m+1} + Y_m + S(-\zeta_m\eta_{m+1}\eta_m), A + 2mQ + S(-\zeta_m)] \\ &= \max[mQ - Z_m + Y_{m+1} + Y_m + S(-\eta_{m+1}\eta_m), \\ & Y_{m+1} + Y_m + S(\zeta_m\eta_{m+1}\eta_m), A + 2mQ + S(\zeta_m)], \\ & \max[Y_{m+1} + S(\eta_{m+1}), Z_m + Z_{m+1} + S(-\zeta_{m+1}\zeta_m)] \\ &= \max[Y_{m+1} + S(-\eta_{m+1}), Z_m + Z_{m+1} + S(\zeta_{m+1}\zeta_m), 0], \end{aligned} \quad (10)$$

which we will show in section 2. The condition that the evolution by the simultaneous equations (9), (10) is unique is written as  $(\zeta_m, Z_m) \neq (+1, mQ)$  and  $(\eta_{m+1}, Y_{m+1}) \neq (+1, 0)$  (see Corollary 1). Moreover, if  $(\zeta_m, Z_m) \neq (+1, mQ)$  and  $(\eta_{m+1}, Y_{m+1}) \neq (+1, 0)$ , then the equations are written in a simpler form, i.e. the amplitude functions satisfy

$$Y_{m+1} + Y_m = A + 2mQ - \max[mQ - Z_m, 0], \quad (11)$$

$$Z_{m+1} + Z_m = \max[Y_{m+1}, 0], \quad (12)$$

and the parity functions satisfy

$$\zeta_{m+1}\zeta_m = \begin{cases} \eta_{m+1} & (Y_{m+1} > 0) \\ -1 & (Y_{m+1} \leq 0) \end{cases}, \quad \eta_{m+1}\eta_m = \begin{cases} \zeta_m & (Z_m < mQ) \\ -1 & (Z_m \geq mQ) \end{cases}. \quad (13)$$

In section 3, we investigate the value of  $(\eta_m, Y_m)$  for the ultradiscrete function  $(\zeta_m, Z_m)$  obtained in [3] and clarify a relationship with the indefinite evolution. We now explain it by the example in Eq.(4). We choose the initial values  $(\zeta_{-17}, Z_{-17}) = (+1, 33)$  and  $(\zeta_{-16}, Z_{-16}) = (+1, -32)$  as the single equation for  $(\zeta_m, Z_m)$ . We introduce the values of  $(\eta_m, Y_m)$  which satisfy the simultaneous equation. It follows from  $Z_{-16} + Z_{-17} = \max[Y_{-16}, 0]$  that  $Y_{-16} = 1$  and also follows from  $\zeta_{-16}\zeta_{-17} = \eta_{-16}$  that  $\eta_{-16} = +1$ . By the evolution determined by Eqs.(9) and (10), we have

$$(\eta_m, Y_m) = \begin{cases} (+1, 0) & (m = -18) \\ (+1, 62) & (m = -17) \\ (+1, 1) & (m = -16) \\ (+1, -6) & (m = -15) \\ (-1, 62) & (m = -14) \\ (-1, -11) & (m = -13) \\ (+1, -1) & (m = -12) \\ (+1, 46) & (m = -11) \\ (+1, -18) & (m = -10) \\ (-1, 11) & (m = -9) \end{cases} \quad (14)$$

and the values  $(\zeta_m, Z_m)$  for  $-18 \leq m \leq -9$  coincides with the one in Eq.(4). In the example, we have unique evolution for  $-18 \leq m \leq -9$ , although indefinite evolution occurs by the values  $(\eta_{-18}, Y_{-18}) = (+1, 0)$  and  $(\zeta_{-9}, Z_{-9}) = (+1, 27)$ .

In the master's thesis of the first author [2], p-ud limit  $(\zeta_m, Z_m)$  of the determinant-type solutions for the case  $a = q^{2M+1}$  and  $M \in \mathbb{Z}_{<0}$  was investigated. We review the results of the p-ud limit and also investigate the value of  $(\eta_m, Y_m)$ .

In section 4, we investigate two parameter solutions of p-ud PII. Note that Murata [8] had investigated two parameter solutions of ultradiscrete PII without parity variables, and our solutions include the patterns which did not appear in [8].

In section 5, we apply the two parameter solutions to obtain the ones which is perturbed from the solutions in section 3. As a perturbation of the solution in Eqs.(4), (14), we investigate the solution whose initial value is given by  $(\eta_{-18}, Y_{-18}) = (+1, 0 - \varepsilon)$ ,  $(\zeta_{-18}, Z_{-18}) = (+1, 29)$  ( $0 < 4\varepsilon < 1$ ). Then the indefiniteness of the solution disappears and the solution of p-ud PII is written as follows:

$$(\eta_m, Y_m), (\zeta_m, Z_m) = \left\{ \begin{array}{lll} \vdots & \vdots & \vdots \\ (-1, 68), & (+1, 36 + \varepsilon), & (m = -20) \\ (-1, 7 + \varepsilon), & (-1, -29), & (m = -19) \\ (+1, 0 - \varepsilon), & (+1, 29), & (m = -18) \\ (+1, 62 + 2\varepsilon), & (+1, 33 + \varepsilon), & (m = -17) \\ (+1, 1), & (+1, -32 - \varepsilon), & (m = -16) \\ (+1, -6 - \varepsilon), & (-1, 32 + \varepsilon), & (m = -15) \\ (-1, 62 - 2\varepsilon), & (+1, 30 + \varepsilon), & (m = -14) \\ (-1, -11 - \varepsilon), & (-1, -30 - \varepsilon), & (m = -13) \\ (+1, -1), & (+1, 30 + \varepsilon), & (m = -12) \\ (+1, 46 + \varepsilon), & (+1, 16), & (m = -11) \\ (+1, -18 - \varepsilon), & (-1, -16), & (m = -10) \\ (-1, 11 + \varepsilon), & (+1, 27 + \varepsilon), & (m = -9) \\ (+1, 22 - \varepsilon), & (+1, -5 - 2\varepsilon), & (m = -8) \\ (+1, -24 - \varepsilon), & (-1, 5 + 2\varepsilon), & (m = -7) \\ \vdots & \vdots & \vdots \end{array} \right. \quad (15)$$

Hence, if  $m \leq -19$  or  $m \geq -8$ , then the values of  $(\zeta_m, Z_m)$  are completely different from Eqs.(4). In other words, the p-ud limits of determinant-type solutions are not stable under the perturbation of initial values.

This paper is organized as follows. In section 2, we derive the simultaneous equation (Eqs.(9), (10)) of p-ud PII and investigate some property of the simultaneous p-ud PII and the single p-ud PII. In section 3, we investigate the values of  $(\eta_m, Y_m)$  for the ultradiscrete function  $(\zeta_m, Z_m)$  obtained in [2, 3] and clarify a relationship with the indefinite evolution. In section 4, we investigate two parameter solutions of p-ud PII. In section 5, we obtain the solutions which are perturbed from the ones in section 3 by using two parameter solutions. In section 6, we give concluding remarks. In the appendix, we review a procedure of obtaining p-ud limit  $(\zeta_m, Z_m)$  of the determinant-type solutions in the case  $a = q^{2M+1}$  and  $M \in \mathbb{Z}_{<0}$ , which is based on [2]. Throughout this paper, we assume  $Q < 0$ .

## 2 Simultaneous ultradiscrete equation with parity variables

We rewrite the simultaneous equation of  $q$ -PII (Eq.(8)) as

$$y_{m+1}y_m = \frac{aq^{2m}z_m}{q^m - z_m}, \quad y_{m+1} = z_{m+1}z_m + 1. \quad (16)$$

We fix a value  $Q(< 0)$  and assume  $0 < q < 1$ . Introduce a variable  $\varepsilon > 0$  by  $q = \exp(Q/\varepsilon)$  and write  $a = \exp(A/\varepsilon)$ . We assume that there exists a one-parameter family of a solution  $y_m(\varepsilon)$ ,  $z_m(\varepsilon)$  ( $\varepsilon$ : positive and sufficiently small) such that

$$y_m(\varepsilon) = \eta_m \exp(Y_m(\varepsilon)/\varepsilon), \quad z_m(\varepsilon) = \zeta_m \exp(Z_m(\varepsilon)/\varepsilon) \quad (17)$$

and the limits  $Y_m(\varepsilon) \rightarrow Y_m$  and  $Z_m(\varepsilon) \rightarrow Z_m$  exist as  $\varepsilon \rightarrow +0$ , where  $\eta_m, \zeta_m \in \{+1, -1\}$  represent the signs of  $y_m(\varepsilon)$ ,  $z_m(\varepsilon)$ . Then we call  $(\eta_m, Y_m)$  (resp.  $(\zeta_m, Z_m)$ ) the  $p$ -ultradiscrete analogue of  $y_m(\varepsilon)$  (resp.  $z_m(\varepsilon)$ ). On the first equation of Eq.(16), we multiply the denominator of the right hand side and substitute Eq.(17) into it. We apply the formulas as

$$\eta_{m+1}\eta_m\zeta_m = \exp(S(\eta_{m+1}\eta_m\zeta_m)/\varepsilon) - \exp(S(-\eta_{m+1}\eta_m\zeta_m)/\varepsilon) \quad (\varepsilon > 0), \quad (18)$$

transpose the negative terms to the other side of equality, and multiply  $e^{-Z_m(\varepsilon)/\varepsilon}$ . Then we have

$$\begin{aligned} & e^{(Y_{m+1}(\varepsilon)+Y_m(\varepsilon)+mQ-Z_m(\varepsilon)+S(\eta_{m+1}\eta_m))/\varepsilon} + e^{(A+2mQ+S(-\zeta_m))/\varepsilon} \\ & + e^{(Y_{m+1}(\varepsilon)+Y_m(\varepsilon)+S(-\eta_{m+1}\eta_m\zeta_m))/\varepsilon} = e^{(Y_{m+1}(\varepsilon)+Y_m(\varepsilon)+mQ-Z_m(\varepsilon)+S(-\eta_{m+1}\eta_m))/\varepsilon} \\ & + e^{(A+2mQ+S(\zeta_m))/\varepsilon} + e^{(Y_{m+1}(\varepsilon)+Y_m(\varepsilon)+S(\eta_{m+1}\eta_m\zeta_m))/\varepsilon}. \end{aligned} \quad (19)$$

It is easy to show that if the limits  $X_j(\varepsilon) \rightarrow X_j$  ( $\varepsilon \rightarrow +0$ ) exist for  $j = 1, \dots, n$ , then

$$\lim_{\varepsilon \rightarrow +0} \varepsilon \log(e^{X_1(\varepsilon)/\varepsilon} + \dots + e^{X_n(\varepsilon)/\varepsilon}) = \max[X_1, \dots, X_n]. \quad (20)$$

Therefore we have the following ultradiscrete equation with parity variables:

$$\begin{aligned} & \max[mQ - Z_m + Y_{m+1} + Y_m + S(\eta_{m+1}\eta_m), \\ & \quad Y_{m+1} + Y_m + S(-\zeta_m\eta_{m+1}\eta_m), A + 2mQ + S(-\zeta_m)] \\ & = \max[mQ - Z_m + Y_{m+1} + Y_m + S(-\eta_{m+1}\eta_m), \\ & \quad Y_{m+1} + Y_m + S(\zeta_m\eta_{m+1}\eta_m), A + 2mQ + S(\zeta_m)], \end{aligned} \quad (21)$$

which coincides with Eq.(9). We also obtain Eq.(10) from the second equation of Eq.(16). It follows from the construction of the  $p$ -ud equation that if  $y_m$  and  $z_m$  are solutions of Eq.(16) and  $(\eta_m, Y_m)$  and  $(\zeta_m, Z_m)$  are the  $p$ -ultradiscrete analogue of  $y_m$  and  $z_m$ , then  $(\eta_m, Y_m)$  and  $(\zeta_m, Z_m)$  satisfy Eqs.(9), (10). Note that if  $y(m)$  is expanded into series of  $q = e^{Q/\varepsilon}$  as

$$y(m) = \hat{\eta}(m)q^{\hat{Y}(m)} \sum_{k=0}^{\infty} d(k, m)q^k, \quad (22)$$

where  $\hat{\eta}(m) \in \{\pm 1\}$  and  $d(0, m) > 0$ , then the p-ultradiscrete analogue of  $y(m)$  is  $(\hat{\eta}(m), \hat{Y}(m)Q)$ .

We investigate uniqueness and indefiniteness of Eqs.(9), (10) and rewrite the equations into a simpler form.

**Proposition 1.** (i) If  $(\zeta_m, Z_m) \neq (+1, mQ)$ , then Eq.(9) is equivalent to

$$Y_{m+1} + Y_m = A + 2mQ - \max[mQ - Z_m, 0], \quad (23)$$

and

$$\eta_{m+1}\eta_m = \begin{cases} \zeta_m & (Z_m < mQ), \\ -1 & (Z_m \geq mQ). \end{cases} \quad (24)$$

(ii) If  $(\zeta_m, Z_m) = (+1, mQ)$ , then Eq.(9) is equivalent to  $Y_m + Y_{m+1} \geq A + 2mQ$ .

(iii) If  $(\eta_{m+1}, Y_{m+1}) \neq (+1, 0)$ , then Eq.(10) is equivalent to

$$Z_m + Z_{m+1} = \max[Y_{m+1}, 0], \quad (25)$$

and

$$\zeta_{m+1}\zeta_m = \begin{cases} \eta_{m+1} & (Y_{m+1} > 0), \\ -1 & (Y_{m+1} \leq 0). \end{cases} \quad (26)$$

(iv) If  $(\eta_{m+1}, Y_{m+1}) = (+1, 0)$ , then Eq.(10) is equivalent to  $Z_m + Z_{m+1} \leq 0$ .

*Proof.* We show (i) and (ii). If  $\zeta_m = -1$ , then it follows from Eq.(9) that

$$\begin{aligned} & \max[mQ - Z_m + Y_{m+1} + Y_m + S(\eta_{m+1}\eta_m), Y_{m+1} + Y_m + S(\eta_{m+1}\eta_m), \\ & \quad A + 2mQ]. \\ & = \max[mQ - Z_m + Y_{m+1} + Y_m + S(-\eta_{m+1}\eta_m), Y_{m+1} + Y_m + S(-\eta_{m+1}\eta_m)]. \end{aligned} \quad (27)$$

Therefore we have  $\eta_{m+1}\eta_m = -1$  and

$$\max[mQ - Z_m + Y_{m+1} + Y_m, Y_{m+1} + Y_m] = A + 2mQ, \quad (28)$$

which is equivalent to Eq.(23).

If  $\zeta_m = +1$ , then it follows from Eq.(9) that

$$\begin{aligned} & \max[mQ - Z_m + Y_{m+1} + Y_m + S(\eta_{m+1}\eta_m), Y_{m+1} + Y_m + S(-\eta_{m+1}\eta_m)] \\ & \quad (29) \\ & = \max[mQ - Z_m + Y_{m+1} + Y_m + S(-\eta_{m+1}\eta_m), Y_{m+1} + Y_m + S(\eta_{m+1}\eta_m), \\ & \quad A + 2mQ]. \end{aligned}$$

If  $mQ - Z_m > 0$  (resp.  $mQ - Z_m < 0$ ), then we have  $\eta_{m+1}\eta_m = +1$  (resp.  $\eta_{m+1}\eta_m = -1$ ) and  $mQ - Z_m + Y_{m+1} + Y_m = A + 2mQ$  (resp.  $Y_{m+1} + Y_m = A + 2mQ$ ). If  $mQ - Z_m = 0$ , then the equation is written as  $Y_{m+1} + Y_m = \max[Y_{m+1} + Y_m, A + 2mQ]$ , which is equivalent to  $Y_{m+1} + Y_m \geq A + 2mQ$ . Therefore we have (i) and (ii).

(iii) and (iv) are shown similarly.  $\square$

Therefore, if  $(\zeta_m, Z_m) \neq (+1, mQ)$  and  $(\eta_{m+1}, Y_{m+1}) \neq (+1, 0)$  and, then the amplitude function satisfies Eqs.(11) and (12).

Assume that the values  $(\eta_{m_0}, Y_{m_0})$  and  $(\zeta_{m_0}, Z_{m_0})$  are given. On the forward evolution,  $(\eta_{m_0+1}, Y_{m_0+1})$  may be determined by Eq.(9) and  $(\zeta_{m_0+1}, Z_{m_0+1})$  may be determined by Eq.(10). On the backward evolution,  $(\zeta_{m_0-1}, Z_{m_0-1})$  may be determined by Eq.(10) and  $(\eta_{m_0-1}, Y_{m_0-1})$  may be determined by Eq.(9). The uniqueness and the indefiniteness for time evolution readily follow from Proposition 1.

**Corollary 1.** (i) If  $(\eta_m, Y_m) \neq (+1, 0)$  (resp.  $(\eta_m, Y_m) = (+1, 0)$ ), then the forward evolution to determine  $(\zeta_m, Z_m)$  is unique (resp. indefinite).  
(ii) If  $(\zeta_m, Z_m) \neq (+1, mQ)$  (resp.  $(\zeta_m, Z_m) = (+1, mQ)$ ), then the forward evolution to determine  $(\eta_{m+1}, Y_{m+1})$  is unique (resp. indefinite).  
(iii) If  $(\zeta_m, Z_m) \neq (+1, mQ)$  (resp.  $(\zeta_m, Z_m) = (+1, mQ)$ ), then the backward evolution to determine  $(\eta_m, Y_m)$  is unique (resp. indefinite).  
(iv) If  $(\eta_m, Y_m) \neq (+1, 0)$  (resp.  $(\eta_m, Y_m) = (+1, 0)$ ), then the backward evolution to determine  $(\zeta_{m-1}, Z_{m-1})$  is unique (resp. indefinite).

We now compare the single p-ud PII (Eq.(2)) with the simultaneous p-ud PII (Eqs.(9), (10)). We investigate the uniqueness and the indefiniteness of the forward evolution of the single p-ud PII.

**Proposition 2.** Set

$$\begin{aligned}\tilde{Z} &= A + 2mQ - \max[Z_{m-1} + Z_m, 0] - \max[mQ - Z_m, 0], \\ Z' &= -Z_m + \max[\tilde{Z}, 0].\end{aligned}\quad (30)$$

Then the forward evolution of Eq.(2) is described as follows:

- (i) Assume that  $(\zeta_{m-1}, \zeta_m) = (+1, +1)$ . If  $mQ - Z_m > 0$  and  $\tilde{Z} \neq 0$ , then  $(\zeta_{m+1}, Z_{m+1}) = (\zeta_m \text{sgn} \tilde{Z}, Z')$ . If  $mQ - Z_m < 0$ , then  $(\zeta_{m+1}, Z_{m+1}) = (-\zeta_m, Z')$ . If  $mQ - Z_m = 0$  or  $(mQ - Z_m > 0$  and  $\tilde{Z} = 0)$ , then the evolution is indefinite.
- (ii) Assume that  $(\zeta_{m-1}, \zeta_m) = (+1, -1)$ . If  $Z_{m-1} + Z_m > 0$  and  $\tilde{Z} \neq 0$ , then  $(\zeta_{m+1}, Z_{m+1}) = (\zeta_m \text{sgn} \tilde{Z}, Z')$ . If  $Z_{m-1} + Z_m < 0$ , then  $(\zeta_{m+1}, Z_{m+1}) = (-\zeta_m, Z')$ . If  $Z_{m-1} + Z_m = 0$  or  $(Z_{m-1} + Z_m > 0$  and  $\tilde{Z} = 0)$ , then the evolution is indefinite.
- (iii) Assume that  $(\zeta_{m-1}, \zeta_m) = (-1, +1)$ . If  $(mQ - Z_m)(Z_{m-1} + Z_m) < 0$  and  $\tilde{Z} \neq 0$ , then  $(\zeta_{m+1}, Z_{m+1}) = (\zeta_m \text{sgn} \tilde{Z}, Z')$ . If  $(mQ - Z_m)(Z_{m-1} + Z_m) > 0$ , then  $(\zeta_{m+1}, Z_{m+1}) = (-\zeta_m, Z')$ . If  $(mQ - Z_m)(Z_{m-1} + Z_m) = 0$  or  $((mQ - Z_m)(Z_{m-1} + Z_m) < 0$  and  $\tilde{Z} = 0)$ , then the evolution is indefinite.
- (iv) Assume that  $(\zeta_{m-1}, \zeta_m) = (-1, -1)$ . Then we have  $(\zeta_{m+1}, Z_{m+1}) = (-\zeta_m, Z')$ .

*Proof.* We show (i). Assume that  $(\zeta_{m-1}, \zeta_m) = (+1, +1)$ . If  $mQ - Z_m > 0$ ,



then Eq.(2) is written as

$$\begin{aligned}
& \max[Z_{m+1} + Z_m + \max[Z_{m-1} + Z_m, 0] + S(\zeta_{m+1}), \\
& \quad Z_{m+1} + mQ + \max[Z_{m-1} + Z_m, 0] + S(-\zeta_{m+1}), A + 2mQ] \\
& = \max[Z_{m+1} + Z_m + \max[Z_{m-1} + Z_m, 0] + S(-\zeta_{m+1}), \\
& \quad Z_{m+1} + mQ + \max[Z_{m-1} + Z_m, 0] + S(\zeta_{m+1}), \\
& \quad \max[Z_{m-1} + Z_m, 0] + mQ - Z_m].
\end{aligned} \tag{31}$$

If  $A + 2mQ > \max[Z_{m-1} + Z_m, 0] + mQ - Z_m$  (i.e.  $\tilde{Z} > 0$ ), then we have  $\zeta_{m+1} = +1$  and  $Z_{m+1} = A + mQ - \max[Z_{m-1} + Z_m, 0]$ , which is equivalent to  $Z_{m+1} = Z'$ . If  $A + 2mQ < \max[Z_{m-1} + Z_m, 0] + mQ - Z_m$  (i.e.  $\tilde{Z} < 0$ ), then we have  $\zeta_{m+1} = -1$  and  $Z_{m+1} = -Z_m$ , which is also equivalent to  $Z_{m+1} = Z'$ . If  $A + 2mQ = \max[Z_{m-1} + Z_m, 0] + mQ - Z_m$ , then we have  $Z_{m+1} \leq A + mQ - \max[Z_{m-1} + Z_m, 0]$ , which implies that the evolution is indefinite.

If  $mQ - Z_m < 0$ , then Eq.(2) is written as

$$\begin{aligned}
& \max[\max[Z_{m-1} + Z_m, 0] + \max[S(\zeta_{m+1}), mQ - Z_m + S(-\zeta_{m+1})] \\
& \quad + Z_{m+1} + Z_m, Z_{m-1} + Z_m, 0, A + 2mQ] \\
& = \max[Z_{m-1} + Z_m, 0] + \max[S(-\zeta_{m+1}), mQ - Z_m + S(\zeta_{m+1})] + Z_{m+1} + Z_m.
\end{aligned} \tag{32}$$

Hence we have  $\zeta_{m+1} = -1$  and  $Z_{m+1} = -Z_m - \max[Z_{m-1} + Z_m, 0] + \max[Z_{m-1} + Z_m, 0, A + 2mQ] = -Z_m + \max[\tilde{Z}, 0]$ .

If  $mQ - Z_m = 0$ , then Eq.(2) is written as

$$\begin{aligned}
& \max[Z_{m+1} + mQ + \max[Z_{m-1} + mQ, 0], Z_{m-1} + mQ, 0, A + 2mQ] \\
& = \max[Z_{m+1} + mQ + \max[Z_{m-1} + mQ, 0], Z_{m-1} + mQ, 0]
\end{aligned} \tag{33}$$

If  $A + 2mQ \leq \max[Z_{m-1} + mQ, 0]$ , then the equation holds for all  $Z_{m+1}$ . Otherwise we have the condition  $Z_{m+1} + mQ + \max[Z_{m-1} + mQ, 0] \geq A + 2mQ$ . Thus the evolution is indefinite.

(ii) (iii) and (iv) are shown similarly.  $\square$

By arranging the previous proposition, we have

**Proposition 3.** If  $(\zeta_{m-1}\zeta_m, Z_{m-1} + Z_m) = (-1, 0)$ ,  $(\zeta_m, Z_m) = (+1, mQ)$ ,  $((\zeta_m, Z_m) = (+1, -A - mQ), Z_{m-1} + Z_m < 0$  and  $mQ - Z_m > 0)$ ,  $((\zeta_{m-1}, Z_{m-1}) = (+1, A + mQ), Z_{m-1} + Z_m > 0$  and  $mQ - Z_m > 0)$  or  $((\zeta_{m-1}\zeta_m, Z_{m-1} + Z_m) = (-1, A + 2mQ), Z_{m-1} + Z_m > 0$  and  $mQ - Z_m < 0)$ , then the forward evolution by Eq.(2) is indefinite. Otherwise, it is unique evolution. Then the amplitude function  $Z_{m+1}$  is written as

$$Z_{m+1} = -Z_m + \max[0, A + 2mQ - \max[0, mQ - Z_m] - \max[0, Z_{m-1} + Z_m]], \tag{34}$$

and the sign function  $\zeta_{m+1}$  is written as

$$\zeta_{m+1} = \begin{cases} -\zeta_m \\ \zeta_m \text{sgn} \tilde{Z} \end{cases} \tag{35}$$

whose conditions are described in Proposition 2.

Note that  $q$ -PII is written as

$$\zeta_{m+1}e^{Z_{m+1}/\varepsilon} = \zeta_m e^{-Z_m/\varepsilon} \cdot \left\{ \frac{e^{(A+2mQ)/\varepsilon}}{(\zeta_{m-1}\zeta_m e^{(Z_{m-1}+Z_m)/\varepsilon} + 1)(\zeta_m e^{(-Z_m+mQ)/\varepsilon} - 1)} - 1 \right\}, \quad (36)$$

and Eq.(34) is interpreted by picking up dominant terms of the right hand side of Eq.(36).

We have similar propositions for the backward evolution. Namely, Propositions 2 and 3 are true for the backward evolution by replacing  $m-1$  (resp.  $m+1$ ) by  $m+1$  (resp.  $m-1$ ). Note that the master thesis of the first author [2] describe the details of the case of the backward evolution.

Under the assumption that a solution  $(\zeta_m, Z_m)$  of the single p-ud PII is given and there exists  $m_0 \in \mathbb{Z}$  such that  $Z_{m_0-1} + Z_{m_0} > 0$ , we construct the function  $(\eta_m, Y_m)$  such that  $(\eta_m, Y_m)$  and  $(\zeta_m, Z_m)$  are a solution of the simultaneous p-ud PII. In order to satisfy  $Z_{m_0-1} + Z_{m_0} = \max[Y_{m_0}, 0]$ , we have  $Y_{m_0} = Z_{m_0-1} + Z_{m_0}$  and it follows from Eq.(13) that  $\eta_{m_0} = \zeta_{m_0-1}\zeta_{m_0}$ . If  $(\zeta_{m_0}, Z_{m_0}) \neq (+1, m_0Q)$ , then it follows from Eq.(11) that  $Y_{m_0+1} = -(Z_{m_0-1} + Z_{m_0}) + A + 2m_0Q - \max[m_0Q - Z_{m_0}, 0]$ , and if  $(\eta_{m_0+1}, Y_{m_0+1}) \neq (+1, 0)$ , then it follows from Eq.(12) that

$$Z_{m_0+1} = -Z_{m_0} + \max[0, A + 2m_0Q - (Z_{m_0-1} + Z_{m_0}) - \max[m_0Q - Z_{m_0}, 0]]. \quad (37)$$

Hence the evolution coincides with Eq.(34) and we have  $Z_{m_0} + Z_{m_0+1} \geq 0$ . We can also show that the sign  $\zeta_{m_0+1}$  coincides with Eq.(35). By repeating the argument, we obtain that if the evolution as the simultaneous equation is unique, then  $(\zeta_{m_0+2}, Z_{m_0+2})$  is written in the form of Eqs.(34), (35) with  $m = m_0 + 1$ . We also obtain that the function  $(\zeta_{m+1}, Z_{m+1})$  also satisfies Eqs.(34), (35) as far as the forward evolution is unique, and it is also true for the backward evolution. Note that the condition  $Z_{m_0-1} + Z_{m_0} \leq 0$  causes the indefinite evolution for single p-ud PII. Namely we have

**Proposition 4.** If  $Z_{m_0-1} + Z_{m_0} \leq 0$ , then indefinite forward evolution or indefinite backward evolution for single p-ud PII occurs around  $m = m_0$ .

*Proof.* Assume that  $Z_{m_0+1}$  is determined uniquely. It follows from  $Z_{m_0-1} + Z_{m_0} \leq 0$  that

$$Z_{m_0+1} = -Z_{m_0} + \max[0, A + 2m_0Q - \max[0, m_0Q - Z_{m_0}]]. \quad (38)$$

The value  $Z_{m_0+1}$  is independent from the value  $Z_{m_0-1} (\leq -Z_{m_0})$ . On the backward evolution such that the values  $Z_{m_0+1}$  and  $Z_{m_0}$  are given, the value  $Z_{m_0-1}$  is determined indefinitely.  $\square$

**Theorem 1.** Assume that a solution of the simultaneous p-ud PII (Eqs.(9), (10)) is given and the function  $(\zeta_m, Z_m)$  also satisfy the single p-ud PII (Eq.(2)). If the evolution by the simultaneous p-ud PII is indefinite at  $m = m'$ , then the evolution by the single p-ud PII is also indefinite around  $m = m'$ .

*Proof.* If the evolution by the simultaneous equation is indefinite at  $m = m'$ , then it follows from Corollary 1 that  $(\zeta_{m''}, Z_{m''}) = (+1, m''Q)$  or  $(\eta_{m''}, Y_{m''}) = (+1, 0)$  for  $m'' = m' - 1, m'$  or  $m' + 1$ . If  $(\zeta_{m''}, Z_{m''}) = (+1, m''Q)$ , then it follows from Proposition 3 that the evolution by Eq.(2) is also indefinite. If  $(\eta_{m''}, Y_{m''}) = (+1, 0)$ , then it follows from Eq.(10) that

$$\max[0, Z_{m''} + Z_{m''-1} + S(-\zeta_{m''-1}\zeta_{m''})] = \max[0, Z_{m''} + Z_{m''-1} + S(\zeta_{m''-1}\zeta_{m''})], \quad (39)$$

which implies  $Z_{m''} + Z_{m''-1} \leq 0$ . By Proposition 4, we have the theorem.  $\square$

On solutions of the single p-ud PII, we have

**Theorem 2.** There exists no solution to the single p-ud PII (Eq.(2)) such that any forward and backward evolution for all  $m \in \mathbb{Z}$  are unique.

*Proof.* Assume that there exists a solution to Eq.(2) such that any forward and backward evolution for all  $m \in \mathbb{Z}$  are unique. Then it follows from Proposition 4 that  $Z_m + Z_{m+1} > 0$  for all  $m \in \mathbb{Z}$ . Since  $Z_{m+1} \neq -Z_m$  and  $Z_{m-1} + Z_m > 0$ , we have

$$Z_{m+1} = -Z_m + A + 2mQ - \max[0, mQ - Z_m] - (Z_{m-1} + Z_m), \quad (40)$$

by Eq.(34). Therefore  $Z_{m+1} + 2Z_m + Z_{m-1} = A + 2mQ$  or  $Z_{m+1} + Z_m + Z_{m-1} = A + mQ$ . If we take the value  $m$  sufficiently large, then  $Z_{m+1} + 2Z_m + Z_{m-1} < 0$  or  $Z_{m+1} + Z_m + Z_{m-1} < 0$ . However it contradicts to  $Z_m + Z_{m+1} > 0$  for all  $m \in \mathbb{Z}$ .  $\square$

### 3 Determinant-type solutions of simultaneous equations

The ultradiscrete limit of determinant-type solutions of  $q$ -PII with a parameter was obtained in [3]. We write it by setting  $C = B - A - (m_0^2 + m_0)Q$  and  $\chi = \alpha\beta$  in [3, Theorem 3].

Assume that  $Q < 0$  and the constant  $A$  in the p-ud PII is written as  $A = (2N + 1)Q$  for  $N \in \mathbb{Z}_{\geq 0}$ . Let  $m_0$  be a negative integer satisfying

$$m_0 \leq \min(-3N - 2, -N(N + 1)/2 - 1), \quad (41)$$

$k_0 \in \{0, 1, \dots, N\}$  and  $C$  be values such that

$$\begin{aligned} & -m_0Q - (N - k_0)(N - k_0 + 1)Q < C \\ & \quad < -m_0Q - (N - k_0 + 1)(N - k_0 + 2)Q (< (m_0 + 1)Q), \quad (k_0 \neq 0), \\ & -m_0Q - N(N - 1)Q < C < (m_0 + 1)Q \quad (k_0 = 0), \end{aligned} \quad (42)$$

and  $\chi \in \{+1, -1\}$ . Using these notation, the following function  $(\zeta^{(N)}(m), Z^{(N)}(m))$  is obtained by the p-ultradiscrete limit of a solution of  $q$ -PII in terms of determinants, and it satisfies p-ud PII (Eq.(2)).

(I) If  $m \leq m_0 - 2N - 1$ , then

$$(\zeta^{(N)}(m), Z^{(N)}(m)) = (+1, (-m - 2N - 1)Q). \quad (43)$$

(II) If  $m_0 - 2N \leq m \leq m_0 + N - 3k_0 + 1$ , then

$$(\zeta^{(N)}(m), Z^{(N)}(m)) = \begin{cases} ((-1)^j \chi, -C - j^2 Q) & (m = m_0 - 2N + 3j) \\ (+1, (m_0 + j + 1)Q) & (m = m_0 - 2N + 3j + 1) \\ ((-1)^j \chi, C + (j + 1)^2 Q) & (m = m_0 - 2N + 3j + 2), \end{cases} \quad (44)$$

where  $0 \leq j \leq N - k_0$  in the first and the second cases and  $0 \leq j \leq N - k_0 - 1$  in the third case.

(III) If  $m = m_0 + N - 3k_0 + 2$  and  $k_0 \neq 0$ , then

$$(\zeta^{(N)}(m), Z^{(N)}(m)) = (-1, (-m_0 - N + k_0 - 1)Q). \quad (45)$$

(IV) If  $m_0 + N - 3k_0 + 3 \leq m \leq m_0 + N$  and  $k_0 \neq 0$ , then

$$(\zeta^{(N)}(m), Z^{(N)}(m)) = \begin{cases} (+1, (m_0 + j)Q) & (m = m_0 - 2N + 3j) \\ ((-1)^j \chi, C + 2m_0 Q + (j + 1)^2 Q) & (m = m_0 - 2N + 3j + 1) \\ ((-1)^{j+1} \chi, -C - 2m_0 Q - (j + 1)^2 Q) & (m = m_0 - 2N + 3j + 2), \end{cases} \quad (46)$$

where  $N - k_0 + 1 \leq j \leq N$  in the first case and  $N - k_0 + 1 \leq j \leq N - 1$  in the second and the third cases.

(V) If  $m_0 + N + 1 \leq m \leq -2N - 1$ , then

$$(\zeta^{(N)}(m), Z^{(N)}(m)) = (+1, mQ). \quad (47)$$

On the function  $(\zeta^{(N)}(m), Z^{(N)}(m))$ , we can confirm the following properties:

(i) If  $k_0 = 0$ , then  $Z^{(N)}(m) < mQ$  for  $m \leq m_0 + N$  and  $Z^{(N)}(m) = mQ$  for  $m = m_0 + N + 1$ .

(ii) If  $k_0 \neq 0$ , then  $Z^{(N)}(m) < mQ$  for  $m \leq m_0 + N - 1$  and  $Z^{(N)}(m) = mQ$  for  $m = m_0 + N$ .

We now calculate the function  $(\eta^{(N)}(m), Y^{(N)}(m))$  associated to the above solution  $(\zeta^{(N)}(m), Z^{(N)}(m))$ .

**Proposition 5.** The function  $(\eta^{(N)}(m), Y^{(N)}(m))$  associated to the solution  $(\zeta^{(N)}(m), Z^{(N)}(m))$  in Eqs.(43)–(47) is written as follows:

(i) If  $m_0 - 2N \leq m \leq m_0 + N - 3k_0 + 1$ , then

$$(\eta^{(N)}(m), Y^{(N)}(m)) = \begin{cases} (+1, 2jQ) & (m = m_0 - 2N + 3j) \\ ((-1)^j \chi, -C + m_0 Q + (-j^2 + j + 1)Q) & (m = m_0 - 2N + 3j + 1) \\ ((-1)^j \chi, C + m_0 Q + (j + 1)(j + 2)Q) & (m = m_0 - 2N + 3j + 2), \end{cases} \quad (48)$$

where  $0 \leq j \leq N - k_0$ .

(ii) If  $m_0 + N - 3k_0 + 3 \leq m \leq m_0 + N$  and  $k_0 \neq 0$ , then

$$\begin{aligned} (\eta^{(N)}(m), Y^{(N)}(m)) = \\ \begin{cases} ((-1)^j \chi, -C - m_0 Q - j(j-1)Q) & (m = m_0 - 2N + 3j) \\ ((-1)^j \chi, C + 3m_0 Q + (j^2 + 3j + 1)Q) & (m = m_0 - 2N + 3j + 1) \\ (+1, 2(j+1)Q) & (m = m_0 - 2N + 3j + 2), \end{cases} \quad (49) \end{aligned}$$

where  $N - k_0 + 1 \leq j \leq N$  in the first case and  $N - k_0 + 1 \leq j \leq N - 1$  in the second and the third cases.

*Proof.* We show (i). Recall that  $Z_{m+1} + Z_m = \max[Y_{m+1}, 0]$  and  $\zeta_{m+1}\zeta_m = \eta_{m+1}$  for  $Y_{m+1} > 0$ . We write  $Z^{(N)}(m)$ ,  $\zeta^{(N)}(m)$ ,  $Y^{(N)}(m)$ ,  $\eta^{(N)}(m)$  as  $Z_m$ ,  $\zeta_m$ ,  $Y_m$ ,  $\eta_m$ . Since  $Z_{m_0-2N+3j} + Z_{m_0-2N+3j+1} = -C - j^2 Q + (m_0 + j + 1)Q > 0$  and  $Z_{m_0-2N+3j+1} + Z_{m_0-2N+3j+2} = C + (j+1)^2 Q + (m_0 + j + 1)Q > 0$ , we have

$$\eta_{m_0-2N+3j+1} = (-1)^j \chi, \quad Y_{m_0-2N+3j+1} = -C - j^2 Q + (m_0 + j + 1)Q, \quad (50)$$

$$\eta_{m_0-2N+3j+2} = (-1)^j \chi, \quad Y_{m_0-2N+3j+2} = C + (j+1)^2 Q + (m_0 + j + 1)Q.$$

It follows from  $Z_m < mQ$  that  $Y_{m_0-2N+3j+1} + Y_{m_0-2N+3j} = (2N+1)Q + (m_0 - 2N + 3j)Q + Z_{m_0-2N+3j}$  and we have  $Y_{m_0-2N+3j} = 2jQ$ . We also have  $\eta_{m_0-2N+3j} = +1$ .

(ii) is shown similarly.  $\square$

On the function  $Y^{(N)}(m)$ , we have  $Y^{(N)}(m_0 - 2N) = 0$  and

$$\begin{aligned} Y^{(N)}(m_0 - 2N + 3j) &< 0, & j = 1, 2, \dots, N - k_0, \\ Y^{(N)}(m_0 - 2N + 3j + 1) &> 0, & j = 0, 1, \dots, N - k_0, \\ Y^{(N)}(m_0 - 2N + 3j + 2) &> 0, & j = 0, 1, \dots, N - k_0 - 1. \end{aligned} \quad (51)$$

If  $k_0 \neq 0$ , then  $Y^{(N)}(m_0 + N - 3k_0 + 2) < 0$ ,  $Y^{(N)}(m_0 + N - 3k_0 + 3) < 0$  and

$$\begin{aligned} Y^{(N)}(m_0 - 2N + 3j) &> 0, & j = N - k_0 + 2, \dots, N, \\ Y^{(N)}(m_0 - 2N + 3j + 1) &> 0, & j = N - k_0 + 1, \dots, N, \\ Y^{(N)}(m_0 - 2N + 3j + 2) &< 0, & j = N - k_0 + 1, \dots, N. \end{aligned} \quad (52)$$

By applying Corollary 1, uniqueness and indefiniteness of the solution of simultaneous p-ud PII can be described.

**Proposition 6.** (i) If  $k_0 \neq 0$ , then we have unique evolution on  $Y^{(N)}(m_0 - 2N)$ ,  $Z^{(N)}(m_0 - 2N)$ ,  $Y^{(N)}(m_0 - 2N + 1)$ ,  $\dots$ ,  $Y^{(N)}(m_0 + N)$ ,  $Z^{(N)}(m_0 + N)$  and indefinite evolution occurs on determining  $Z^{(N)}(m_0 - 2N - 1)$  and  $Y^{(N)}(m_0 + N + 1)$ .

(ii) If  $k_0 = 0$ , then we have unique evolution on  $Y^{(N)}(m_0 - 2N)$ ,  $Z^{(N)}(m_0 - 2N)$ ,  $Y^{(N)}(m_0 - 2N + 1)$ ,  $\dots$ ,  $Y^{(N)}(m_0 + N + 1)$ ,  $Z^{(N)}(m_0 + N + 1)$  and indefinite evolution occurs on determining  $Z^{(N)}(m_0 - 2N - 1)$  and  $Y^{(N)}(m_0 + N + 2)$ .

In [2], the ultradiscrete limit of determinant-type solutions of  $q$ -PII for the case  $Q < 0$  and  $A = (2M + 1)Q$ ,  $M \in \mathbb{Z}_{\leq -1}$  was obtained, and we review it in the appendix. We describe the explicit form of the ultradiscrete limit.

Let  $m_0$  be a value satisfying

$$m_0 \leq \min(3M + 1, -M(M + 1)/2 - 1), \quad (53)$$

$k_0 \in \{0, 1, \dots, -M - 1\}$  and  $C$  be values such that

$$\begin{aligned} -m_0Q - (M + k_0 + 1)(M + k_0)Q &\leq C \\ &< -m_0Q - (M + k_0)(M + k_0 - 1)Q, \quad k_0 \neq 0, \\ -m_0Q - M(M + 1)Q &< C < (m_0 + 1)Q, \quad k_0 = 0, \end{aligned} \quad (54)$$

and  $\chi \in \{+1, -1\}$ . Using these notation, the following function  $(\zeta^{(M)}(m), Z^{(M)}(m))$  is obtained by setting  $C = B - A - (m_0^2 + m_0)Q$  and  $\chi = \alpha\beta$  in Theorem 3, and it satisfies p-ud PII (Eq.(2)).

**Proposition 7.** (i) If  $m \leq m_0 + M$ , then

$$(\zeta^{(M)}(m), Z^{(M)}(m)) = (+1, mQ). \quad (55)$$

(ii) If  $m_0 + M + 1 \leq m \leq m_0 - 2M - 3k_0$  ( $k_0 \neq 0$ ) or  $m_0 + M + 1 \leq m \leq m_0 - 2M - 1$  ( $k_0 = 0$ ), then

$$\begin{aligned} &(\zeta^{(M)}(m), Z^{(M)}(m)) = \\ &\begin{cases} (+1, (m_0 + M + j)Q) & (m = m_0 + M + 3j - 2) \\ ((-1)^{j-1}\chi, C + (j^2 + M)Q) & (m = m_0 + M + 3j - 1) \\ ((-1)^{j-1}\chi, -C + (-j^2 + 2j + M)Q) & (m = m_0 + M + 3j), \end{cases} \end{aligned} \quad (56)$$

( $1 \leq j \leq -M - k_0$ ). The case  $k_0 = 0$  and  $m = m_0 - 2M$  is included in (iv).

(iii) If  $m_0 - 2M - 3k_0 + 1 \leq m \leq m_0 - 2M - 2$  and  $k_0 \neq 0$ , then

$$\begin{aligned} &(\zeta^{(M)}(m), Z^{(M)}(m)) = \\ &\begin{cases} ((-1)^{j-1}\chi, C + (2m_0 + j^2 + M)Q) & (m = m_0 + M + 3j - 2) \\ ((-1)^{j-1}\chi, -C + (-2m_0 - j^2 + 2j + M)Q) & (m = m_0 + M + 3j - 1) \\ (+1, (m_0 + M + j)Q) & (m = m_0 + M + 3j). \end{cases} \end{aligned} \quad (57)$$

Here  $-M - k_0 + 1 \leq j \leq -M$  for the first case and  $-M - k_0 + 1 \leq j \leq -M - 1$  for the second and the third cases.

(iv) If  $m_0 - 2M - 1 \leq m \leq M + 1$  ( $k_0 \neq 0$ ) or  $m_0 - 2M \leq m \leq M + 1$  ( $k_0 = 0$ ), then

$$(\zeta^{(M)}(m), Z^{(M)}(m)) = (+1, (-m - 2M - 1)Q). \quad (58)$$

The function  $Z^{(M)}(m)$  in Proposition 7 satisfies  $Z^{(M)}(m) = mQ$  for  $m \leq m_0 + M + 1$  and

$$\begin{aligned} Z^{(M)}(m) &> mQ, \quad m = m_0 + M + 3j - 2, \quad j = 2, 3, \dots, -M - k_0, \\ Z^{(M)}(m) &< mQ, \quad m = m_0 + M + 3j - 1, \quad j = 1, 2, \dots, -M - k_0, \\ Z^{(M)}(m) &< mQ, \quad m = m_0 + M + 3j, \quad j = 1, 2, \dots, -M - k_0 - 1. \end{aligned} \quad (59)$$

If  $k_0 \neq 0$ , then  $Z^{(M)}(m) > mQ$  for  $m = m_0 - 2M - 3k_0$ ,  $m_0 - 2M - 3k_0 + 1$ , and

$$\begin{aligned} Z^{(M)}(m) &< mQ, & m = m_0 + M + 3j - 2, & j = -M - k_0 + 2, \dots, -M, \\ Z^{(M)}(m) &< mQ, & m = m_0 + M + 3j - 1, & j = -M - k_0 + 1, \dots, -M, \\ Z^{(M)}(m) &> mQ, & m = m_0 + M + 3j, & j = -M - k_0 + 1, \dots, -M. \end{aligned} \quad (60)$$

We now calculate the function  $(\eta^{(M)}(m), Y^{(M)}(m))$  associated to the above solution  $(\zeta^{(M)}(m), Z^{(M)}(m))$ .

**Proposition 8.** The function  $(\eta^{(M)}(m), Y^{(M)}(m))$  associated to the function  $(\zeta^{(M)}(m), Z^{(M)}(m))$  in Proposition 7 is written as follows:

(i) If  $m_0 + M + 2 \leq m \leq m_0 - 2M - 3k_0$ , then

$$\begin{aligned} (\eta^{(M)}(m), Y^{(M)}(m)) = & \\ \begin{cases} ((-1)^{j-1}\chi, C + m_0Q + 2MQ + j(j+1)Q) & (m = m_0 + M + 3j - 1) \\ (+1, 2(M+j)Q) & (m = m_0 + M + 3j) \\ ((-1)^{j-1}\chi, -C + m_0Q + 2MQ + (-j^2 + 3j + 1)Q) & (m = m_0 + M + 3j + 1), \end{cases} \end{aligned} \quad (61)$$

where  $1 \leq j \leq -M - k_0$  in the first and the second cases and  $0 \leq j \leq -M - k_0 - 1$  in the third case.

(ii) If  $k_0 \neq 0$ , then

$$\begin{aligned} (\eta^{(M)}(m), Y^{(M)}(m)) = & (-1, (2m_0 - 2M - 4k_0 + 1)Q) \\ & (m = m_0 - 2M - 3k_0 + 1). \end{aligned} \quad (62)$$

(iii) If  $m_0 - 2M - 3k_0 + 2 \leq m \leq m_0 - 2M - 1$  and  $k_0 \neq 0$ , then

$$\begin{aligned} (\eta^{(M)}(m), Y^{(M)}(m)) = & \\ \begin{cases} (+1, 2(M+j)Q) & (m = m_0 + M + 3j - 1) \\ ((-1)^{j-1}\chi, -C + (-m_0 + 2M - j^2 + 3j)Q) & (m = m_0 + M + 3j) \\ ((-1)^{j-1}\chi, C + (3m_0 + 2M + j^2 + 3j + 1)Q) & (m = m_0 + M + 3j + 1), \end{cases} \end{aligned} \quad (63)$$

where  $-M - k_0 + 1 \leq j \leq -M$  for the first case and the second cases, and  $-M - k_0 + 1 \leq j \leq -M - 1$  for the third case.

*Proof.* (i) and (iii) are shown by using  $Z_{m+1} + Z_m > 0$ ,  $Z_{m+1} + Z_m = \max[Y_{m+1}, 0]$  and  $\zeta_{m+1}\zeta_m = \eta_{m+1}$  for  $Y_{m+1} > 0$ . We show (ii). In the case  $j = -M - k_0$ , we have  $Z_{m_0+M+3j} - (m_0+M+3j)Q = -C - m_0Q + (M+k_0)(M+k_0-1)Q > 0$ . By applying evolution of p-ud PII, we have  $Y_{m_0-2M-3k_0+1} = -Y_{m_0+M+3(-M-k_0)} + (2M+1)Q + 2(m_0 - 2M - 3k_0 + 1)Q$  and  $\eta_{m_0-2M-3k_0+1} = -\eta_{m_0+M+3(-M-k_0)}$ . Therefore we have (ii).  $\square$

If  $k_0 = 0$ , then  $Y^{(M)}(m) > 0$  for  $m_0 + M + 1 \leq m \leq m_0 - 2M - 1$  and  $Y^{(M)}(m) = 0$  for  $m = m_0 - 2M$ . If  $k_0 \neq 0$ , then  $Y^{(M)}(m) > 0$  for  $m_0 + M + 1 \leq m \leq m_0 - 2M - 2$  and  $Y^{(M)}(m) = 0$  for  $m = m_0 - 2M - 1$ .

**Proposition 9.** (i) If  $k_0 \neq 0$ , then we have unique evolution on  $Z^{(M)}(m_0 + M + 1), Y^{(M)}(m_0 + M + 2), Z^{(M)}(m_0 + M + 2), \dots, Z^{(M)}(m_0 - 2M - 2), Y^{(M)}(m_0 - 2M - 1)$  and indefinite evolution occurs on determining  $Y^{(M)}(m_0 + M + 1)$  and  $Z^{(M)}(m_0 - 2M - 1)$ .

(ii) If  $k_0 = 0$ , then we have unique evolution on  $Z^{(M)}(m_0 + M + 1), Y^{(M)}(m_0 + M + 2), Z^{(M)}(m_0 + M + 2), \dots, Z^{(M)}(m_0 - 2M - 1), Y^{(M)}(m_0 - 2M)$  and indefinite evolution occurs on determining  $Y^{(M)}(m_0 + M + 1)$  and  $Z^{(M)}(m_0 - 2M)$ .

We show an example. If  $M = -3, Q = -2, C = -10, k_0 = 1, \chi = +1$ , then the function  $(\zeta_m, Z_m)$  in Proposition 7 is written as

$$(\zeta_m, Z_m) = \begin{cases} (+1, -2m) & (m \leq -13) \\ (+1, 24) & (m = -12) \\ (+1, -6) & (m = -11) \\ (+1, 14) & (m = -10) \\ (+1, 22) & (m = -9) \\ (-1, -12) & (m = -8) \\ (-1, 16) & (m = -7) \\ (+1, 18) & (m = -6) \\ (+1, 2m - 10) & (-5 \leq m \leq -1). \end{cases} \quad (64)$$

Note that we fixed an error in [2]. It follows from Proposition 8 that

$$(\eta_m, Y_m) = \begin{cases} (+1, 18) & (m = -11) \\ (+1, 8) & (m = -10) \\ (+1, 36) & (m = -9) \\ (-1, 10) & (m = -8) \\ (+1, 4) & (m = -7) \\ (-1, 34) & (m = -6) \\ (+1, 0) & (m = -5). \end{cases} \quad (65)$$

## 4 Two parameter solutions

In this section, we investigate two parameter solutions of the simultaneous p-ud PII (Eqs.(9), (10)). Firstly we investigate solutions under the condition

$$Y_m < 0, Z_m > mQ \quad (m = m', m' + 1, \dots). \quad (66)$$

Then it follows from Eqs.(11)–(13) that the simultaneous p-ud PII is written as

$$\begin{aligned} Y_{m+1} + Y_m &= A + 2mQ, \quad Z_{m+1} + Z_m = 0, \\ \eta_{m+1}\eta_m &= -1, \quad \zeta_{m+1}\zeta_m = -1. \end{aligned} \quad (67)$$

By setting  $Z_{m'} = d_1, Y_{m'} = ((2m' - 1)Q + A)/2 + d_2, \zeta_{m'} = \zeta, \eta_{m'} = \eta$ , it is solved as

$$\begin{aligned} (\eta_m, Y_m) &= ((-1)^{m-m'}\eta, \frac{(2m-1)Q + A}{2} + d_2(-1)^{m-m'}), \\ (\zeta_m, Z_m) &= ((-1)^{m-m'}\zeta, d_1(-1)^{m-m'}), \end{aligned} \quad (68)$$



and satisfies  $Y_{m'+2} = Y_{m'} + 2Q$ ,  $Z_{m'+2} = Z_{m'}$ . Namely it has 2-periodic structure. By combining with the condition  $Y_m < 0$  and  $Z_m > mQ$ , we have the following proposition:

**Proposition 10.** (i) Let  $d_1$  and  $d_2$  be real constants and  $\eta, \zeta \in \{\pm 1\}$ . If  $Q < 0$  and the integer  $m'$  satisfies

$$\begin{aligned} m'Q &< \min(d_1, -d_1 - Q), \\ 2m'Q &< -Q - A + \min(d_2, -d_2 - 2Q), \end{aligned} \quad (69)$$

then the functions  $(\eta_m, Y_m)$ ,  $(\zeta_m, Z_m)$  defined by Eq.(68) satisfy p-ud PII (Eqs.(9), (10)) for  $m \geq m'$ .

(ii) If  $Q < 0$  and there exists a integer  $m'$  such that the functions  $(\eta_m, Y_m)$ ,  $(\zeta_m, Z_m)$  satisfy p-ud PII (Eqs.(9), (10)) for  $m \geq m'$  and  $Y_m < 0$  and  $Z_m > mQ$  for  $m = m', m' + 1$ , then they are written in the form of Eq.(68) and satisfy  $Y_m < 0$  and  $Z_m > mQ$  for  $m \geq m'$ .

*Proof.* (i) If  $m \geq m'$ , then it follows from Eq.(69) that the functions  $Y_m$  and  $Z_m$  in Eq.(68) satisfy  $Y_m < 0$  and  $Z_m > mQ$ . We can confirm Eq.(67) directly.

(ii) It follows from the assumption that  $Y_{m'+2} = Y_{m'} + 2Q$  and  $Z_{m'+2} = Z_{m'}$ . Hence we have  $Y_{m'+2} < 2Q < 0$ ,  $Z_{m'+2} > m'Q > (m' + 2)Q$ , Eq.(67) for  $m = m' + 2$  and  $Y_{m'+3} = Y_{m'+1} + 2Q < 0$ . Similarly we have  $Z_{m'+3} > (m' + 3)Q$ , Eq.(67) for  $m = m' + 3$  and  $Y_{m'+4} = Y_{m'+2} + 2Q < 0$ . Thus we have (ii) inductively.  $\square$

Note that the solution in the form of Eq.(68) was essentially obtained by Murata [8] for the case of ultradiscrete PII without parity variables.

Next we investigate solutions of p-ud PII under the condition

$$Y_m > 0, Z_m < mQ \quad (m = m', m' - 1, \dots). \quad (70)$$

Then we have

$$\begin{aligned} Z_{m-1} + Z_m &= Y_m, Y_{m-1} + Y_m = Z_{m-1} + A + (m-1)Q, \\ \zeta_{m-1}\zeta_m &= \eta_m, \eta_{m-1}\eta_m = \zeta_{m-1}. \end{aligned} \quad (71)$$

By setting  $Z_{m'} = C + (m'Q + A)/3$  and  $Y_{m'} = -D + ((2m' - 1)Q + 2A)/3$ ,  $\zeta_{m'} = \zeta$ ,  $\eta_{m'} = \eta$ , we have

$$\begin{aligned} Z_{m'-1} &= Y_{m'} - Z_{m'} = \frac{(m'-1)Q + A}{3} - C - D, \\ Y_{m'-1} &= -Y_{m'} + Z_{m'-1} + A + (m'-1)Q = \frac{(2m'-3)Q + 2A}{3} - C, \\ Z_{m'-2} &= \frac{(m'-2)Q + A}{3} + D, Y_{m'-2} = \frac{(2m'-5)Q + 2A}{3} + C + D, \\ Z_{m'-3} &= \frac{(m'-3)Q + A}{3} + C, Y_{m'-3} = \frac{(2m'-7)Q + 2A}{3} - D. \end{aligned} \quad (72)$$

Thus

$$Y_{m'-3} = Y_{m'} - 2Q, \quad Z_{m'-3} = Z_{m'} - Q, \quad (73)$$

i.e. the solution has 3-periodic structure. By combining with the condition  $Y_m > 0$  and  $Z_m < mQ$  and discussion on parity variables, we have the following proposition:

**Proposition 11.** (i) Let  $c_m$  be real sequence such that  $c_{m-2} + c_{m-1} + c_m = 0$  and  $\eta, \zeta \in \{\pm 1\}$ . If  $Q < 0$  and the integer  $m'$  satisfies

$$\begin{aligned} 2m'Q &> Q - 2A + \max(3c_{m'+1}, 3c_{m'} + 2Q, 3c_{m'-1} + 4Q), \\ 2m'Q &> 2Q + A + \max(3c_{m'-1}, 3c_{m'+1} + 2Q, 3c_{m'} + 4Q), \end{aligned} \quad (74)$$

then the functions  $(\zeta_m, Z_m), (\eta_m, Y_m)$  defined by

$$\begin{aligned} Z_m &= \frac{mQ + A}{3} + c_m, \quad Y_m = \frac{(2m-1)Q + 2A}{3} - c_{m+1}, \\ \zeta_{m'-3k} &= \zeta, \quad \eta_{m'-3k} = \eta, \quad \zeta_{m'-3k-1} = \zeta\eta, \\ \eta_{m'-3k-1} &= \zeta, \quad \zeta_{m'-3k-2} = \eta, \quad \eta_{m'-3k-2} = \zeta\eta, \quad (k \in \mathbb{Z}_{\geq 0}) \end{aligned} \quad (75)$$

satisfy p-ud PII (Eqs.(9), (10)) for  $m \leq m' - 1$ .

(ii) If  $Q < 0$  and there exists a integer  $m'$  such that the functions  $(\zeta_m, Z_m), (\eta_m, Y_m)$  satisfy p-ud PII (Eqs.(9), (10)) for  $m \leq m' - 1$  and  $Y_m > 0$  and  $Z_m < mQ$  for  $m = m', m' - 1, m' - 2$ , then the solution is written as Eq.(75) and we have  $Y_m > 0$  and  $Z_m < mQ$  for  $m \leq m'$ .

*Proof.* It is proved similarly to Proposition 10 by applying backward evolution.  $\square$

Note that the solution in the form of Eq.(75) was essentially obtained by Murata [8], and the condition  $c_{m-2} + c_{m-1} + c_m = 0$  for  $m \leq m'$  implies that  $c_{m-3} = c_m$  for  $m \leq m'$ .

We are going to find solutions which are close to the one in Proposition 11. If the condition  $Y_m > 0$  and  $Z_m < mQ$  for  $m = m', m' - 1, m' - 2$  is satisfied, then the solution of p-ud PII for  $m \leq m' - 1$  is determined as Proposition 11. We investigate two cases (Eqs.(76) and (79)) which are close to the case in Proposition 11. Let  $K \in \mathbb{Z}_{\geq 0}$ . In the case

$$\begin{aligned} Z_m &< mQ, \quad (m' \leq m \leq m' + 3K + 2), \\ Y_{m'+3k+1} &> 0, \quad Y_{m'+3k+2} > 0, \quad Y_{m'+3k+3} < 0, \quad (k = 0, 1, \dots, K), \end{aligned} \quad (76)$$

we have

$$\begin{aligned} Y_{m+1} + Y_m &= Z_m + A + mQ, \quad (m' \leq m \leq m' + 3K + 2), \\ Z_{m+1} + Z_m &= Y_{m+1}, \quad (m = m' + 3k + 1, m' + 3k + 2, k = 0, 1, \dots, K), \\ Z_{m+1} + Z_m &= 0, \quad (m = m' + 3k + 3, k = 0, 1, \dots, K), \end{aligned} \quad (77)$$

and the corresponding equations for the parity variables. Let  $C', D'$  be constants and  $\eta, \zeta \in \{\pm 1\}$ . The function

$$\begin{aligned}
(\eta_{m'+3k'}, Y_{m'+3k'}) &= (\eta, 2k'Q + D'), \\
(\zeta_{m'+3k'}, Z_{m'+3k'}) &= (\zeta(-\eta)^{k'}, -(k')^2Q - C' - k'D'), \\
(\eta_{m'+3k+1}, Y_{m'+3k+1}) &= (\eta\zeta(-\eta)^k, (m' - k^2 + k)Q + A - C' - (k+1)D'), \\
(\zeta_{m'+3k+1}, Z_{m'+3k+1}) &= (\eta, (m' + k)Q + A - D'), \\
(\eta_{m'+3k+2}, Y_{m'+3k+2}) &= (\zeta(-\eta)^k, m' + k^2 + 3k + 1)Q + A + C' + kD'), \\
(\zeta_{m'+3k+2}, Z_{m'+3k+2}) &= (\eta\zeta(-\eta)^k, (k+1)^2Q + C' + (k+1)D'),
\end{aligned} \tag{78}$$

for  $k = 0, 1, \dots, K$  and  $k' = 0, 1, \dots, K+1$  satisfies p-ud PII, if the constants  $C'$  and  $D'$  satisfy the condition in Eq.(76).

Next we investigate the case

$$\begin{aligned}
Y_m &> 0, \quad (m' + 1 \leq m \leq m' + 3K + 3) \\
Z_{m'+3k+1} &< (m' + 3k + 1)Q, \quad Z_{m'+3k+2} < (m' + 3k + 2)Q, \\
Z_{m'+3k+3} &> (m' + 3k + 3)Q, \quad (k = 0, 1, \dots, K).
\end{aligned} \tag{79}$$

The function

$$\begin{aligned}
(\zeta_{m'+3k'}, Z_{m'+3k'}) &= (\zeta, (m' + k')Q + D'), \\
(\eta_{m'+3k'+1}, Y_{m'+3k'+1}) &= (-\eta(-\zeta)^{k'}, (m' + k'(k' + 3))Q - k'D' + C'), \\
(\zeta_{m'+3k+1}, Z_{m'+3k+1}) &= (-\eta\zeta(-\zeta)^k, k(k+2)Q - (k+1)D' + C'), \\
(\eta_{m'+3k+2}, Y_{m'+3k+2}) &= (\zeta, (2k+1)Q + A - D'), \\
(\zeta_{m'+3k+2}, Z_{m'+3k+2}) &= (-\eta(-\zeta)^k, (-k^2 + 1)Q + A + kD' - C'), \\
(\eta_{m'+3k+3}, Y_{m'+3k+3}) &= (\eta(-\zeta)^{k+1}, (m' - (k+1)(k-2))Q + A + (k+1)D' - C'),
\end{aligned} \tag{80}$$

for  $k = 0, 1, \dots, K$  and  $k' = 0, 1, \dots, K+1$  satisfies p-ud PII, if the constants  $C'$  and  $D'$  satisfy the condition in Eq.(79).

We are going to find patterns of solutions which are close to the one in Proposition 10. If the condition  $Y_m < 0$  and  $Z_m > mQ$  for  $m = m', m' + 1$  is satisfied, then the solution of p-ud PII for  $m \geq m'$  is determined as Proposition 10. We investigate two cases (Eqs.(81) and (83)) which are close to the case in Proposition 10. First we investigate the case

$$\begin{aligned}
Y_m &< 0, \quad (m' + 1 \leq m \leq m' + 2K + 2), \\
Z_{m'+2k} &< (m' + 2k)Q, \quad Z_{m'+2k+1} > (m' + 2k + 1)Q, \quad (k = 0, 1, \dots, K).
\end{aligned} \tag{81}$$

The function

$$\begin{aligned}
(\eta_{m'+2k'}, Y_{m'+2k'}) &= (\eta(-\zeta)^{k'}, k'(m' + k' + 1)Q + k'D' + C'), \\
(\zeta_{m'+2k'}, Z_{m'+2k'}) &= (\zeta, -D'), \\
(\eta_{m'+2k+1}, Y_{m'+2k+1}) &= (\zeta\eta(-\zeta)^k, -(k-1)(m' + k)Q + A - (k+1)D' - C'), \\
(\zeta_{m'+2k+1}, Z_{m'+2k+1}) &= (-\zeta, D'),
\end{aligned} \tag{82}$$

for  $k = 0, 1, \dots, K$  and  $k' = 0, 1, \dots, K + 1$  satisfies p-ud PII, if the constants  $C'$  and  $D'$  satisfy the condition in Eq.(81). Next we investigate the case

$$\begin{aligned} Z_m &> mQ, \quad (m' \leq m \leq m' + 2K + 1), \\ Y_{m'+2k+1} &< 0, \quad Y_{m'+2k+2} > 0, \quad (k = 0, 1, \dots, K). \end{aligned} \quad (83)$$

The function

$$\begin{aligned} (\eta_{m'+2k'}, Y_{m'+2k'}) &= (\eta, 2k'Q + D'), \\ (\zeta_{m'+2k'}, Z_{m'+2k'}) &= (\zeta(-\eta)^{k'}, -(k' - 1)k'Q + C' - k'D'), \\ (\eta_{m'+2k+1}, Y_{m'+2k+1}) &= (-\eta, 2(m' + k)Q + A - D'), \\ (\zeta_{m'+2k+1}, Z_{m'+2k+1}) &= (-\zeta(-\eta)^k, k(k + 1)Q - C' + (k + 1)D'), \end{aligned} \quad (84)$$

for  $k = 0, 1, \dots, K$  and  $k' = 0, 1, \dots, K + 1$  satisfies p-ud PII, if the constants  $C'$  and  $D'$  satisfy the condition in Eq.(83).

## 5 Perturbed solutions

We investigate solutions of p-ud PII which are close to the functions  $(\eta^{(N)}(m), Y^{(N)}(m))$ ,  $(\zeta^{(N)}(m), Z^{(N)}(m))$  and  $(\eta^{(M)}(m), Y^{(M)}(m))$ ,  $(\zeta^{(M)}(m), Z^{(M)}(m))$  in section 3.

Let us recall the situation that the function  $(\zeta^{(N)}(m), Z^{(N)}(m))$  in Eqs.(43)–(47) was described. Assume that the constant  $A$  in the p-ud PII is written as  $A = (2N + 1)Q$ ,  $N \in \mathbb{Z}_{\geq 1}$ ,  $k_0 \in \{0, 1, \dots, N\}$ ,  $\chi \in \{+1, -1\}$  and the value  $m_0 \in \mathbb{Z}_{< 0}$  satisfies Eqs.(41). Let  $\varepsilon$  be a real number such that  $|\varepsilon|$  is sufficiently small. By setting  $A = (2N + 1)Q$ ,  $m' = m_0 - 2N$ ,  $k = j$ ,  $C' = C$ ,  $D' = \varepsilon$ ,  $\eta = +1$  and  $\zeta = \chi$  in Eq.(78), we have

$$\begin{aligned} (\eta_{m_0-2N+3j}, Y_{m_0-2N+3j}) &= (+1, 2jQ + \varepsilon), \\ (\zeta_{m_0-2N+3j}, Z_{m_0-2N+3j}) &= ((-1)^j \chi, -j^2 Q - C - j\varepsilon), \\ (\eta_{m_0-2N+3j+1}, Y_{m_0-2N+3j+1}) &= ((-1)^j \chi, (m_0 - j^2 + j + 1)Q - C - (j + 1)\varepsilon), \\ (\zeta_{m_0-2N+3j+1}, Z_{m_0-2N+3j+1}) &= (+1, (m_0 + j + 1)Q - \varepsilon), \\ (\eta_{m_0-2N+3j+2}, Y_{m_0-2N+3j+2}) &= ((-1)^j \chi, (m_0 + j^2 + 3j + 2)Q + C + j\varepsilon), \\ (\zeta_{m_0-2N+3j+2}, Z_{m_0-2N+3j+2}) &= ((-1)^j \chi, (j + 1)^2 Q + C + (j + 1)\varepsilon). \end{aligned} \quad (85)$$

We discuss the case  $k_0 = 0$  with the condition  $-m_0 Q - N(N - 1)Q < C < (m_0 + 1)Q$ . Then the function in Eq.(85) satisfies

$$Z_m < mQ, \quad Y_{m_0-2N+3j+1} > 0, \quad Y_{m_0-2N+3j'+2} > 0, \quad Y_{m_0-2N+3j'+3} < 0 \quad (86)$$

for  $m_0 - 2N \leq m \leq m_0 + N$ ,  $0 \leq j \leq N$  and  $0 \leq j' \leq N - 1$ . Hence the function  $(\eta_m, Y_m)$  for  $m_0 - 2N \leq m \leq m_0 + N + 1$  and  $(\zeta_m, Z_m)$  for  $m_0 - 2N \leq m \leq m_0 + N + 1$  in Eq.(85) satisfies p-ud PII. If  $\varepsilon = 0$ , then the function defined by Eq.(85) coincides with the solution  $(\eta^{(N)}(m), Y^{(N)}(m))$ ,  $(\zeta^{(N)}(m), Z^{(N)}(m))$  in section 3, which has indefinite evolution caused by  $(\eta_{m_0-2N}, Y_{m_0-2N}) = (+1, 0)$

and  $(\zeta_{m_0+N+1}, Z_{m_0+N+1}) = (+1, (m_0 + N + 1)Q)$ . By imposing  $\varepsilon \neq 0$ , the indefiniteness of the evolution may disappear. We now investigate the backward evolution for  $m \leq m_0 - 2N$ . Recall that

$$\begin{aligned}(\zeta_{m_0-2N}, Z_{m_0-2N}) &= (\chi, -C), \\ (\eta_{m_0-2N}, Y_{m_0-2N}) &= (+1, \varepsilon),\end{aligned}\tag{87}$$

If  $\varepsilon < 0$ , then  $Y_{m_0-2N} = \varepsilon < 0$  and we have

$$\begin{aligned}(\zeta_{m_0-2N-1}, Z_{m_0-2N-1}) &= (-\chi, C), \\ (\eta_{m_0-2N-1}, Y_{m_0-2N-1}) &= (-\chi, m_0Q + C - \varepsilon), \\ (\zeta_{m_0-2N-2}, Z_{m_0-2N-2}) &= (+1, m_0Q - \varepsilon), \\ (\eta_{m_0-2N-2}, Y_{m_0-2N-2}) &= (-\chi, (m_0 - 1)Q - C), \\ (\zeta_{m_0-2N-3}, Z_{m_0-2N-3}) &= (-\chi, -Q - C + \varepsilon), \\ (\eta_{m_0-2N-3}, Y_{m_0-2N-3}) &= (+1, -2Q + \varepsilon),\end{aligned}\tag{88}$$

and the conditions  $Z_m < mQ$  and  $Y_m > 0$  are satisfied for  $m = m_0 - 2N - 1, m_0 - 2N - 2, m_0 - 2N - 3$ . Then we can use Proposition 11 (ii), and the solution for  $m \leq m_0 - 2N - 1$  is expressed in the form of Proposition 11 (i) by setting  $c_{m_0} = -C + \varepsilon - (m_0 + 1)Q/3$ ,  $c_{m_0+1} = -\varepsilon + (2m_0 + 1)Q/3$ ,  $c_{m_0+2} = C - m_0Q/3$ . Namely if  $j < 0$ , then we have

$$\begin{aligned}(\eta_{m_0-2N+3j}, Y_{m_0-2N+3j}) &= (+1, 2jQ + \varepsilon), \\ (\zeta_{m_0-2N+3j}, Z_{m_0-2N+3j}) &= (-\chi, jQ - C + \varepsilon), \\ (\eta_{m_0-2N+3j+1}, Y_{m_0-2N+3j+1}) &= (-\chi, (m_0 + 2j + 1)Q - C), \\ (\zeta_{m_0-2N+3j+1}, Z_{m_0-2N+3j+1}) &= (+1, (m_0 + j + 1)Q - \varepsilon), \\ (\eta_{m_0-2N+3j+2}, Y_{m_0-2N+3j+2}) &= (-\chi, (m_0 + 2j + 2)Q + C - \varepsilon), \\ (\zeta_{m_0-2N+3j+2}, Z_{m_0-2N+3j+2}) &= (-\chi, (j + 1)Q + C).\end{aligned}\tag{89}$$

If  $\varepsilon > 0$ , then we have  $Y_{m_0-2N} = \varepsilon > 0$  and the conditions  $Z_m < mQ$  and  $Y_m > 0$  are satisfied for  $m = m_0 - 2N + 2, m_0 - 2N + 1, m_0 - 2N$ . Then we can use Proposition 11 (ii) and the solution for  $m \leq m_0 - 2N + 2$  is expressed in the form of Proposition 11 (i) by setting  $c_{m_0} = -C - (m_0 + 1)Q/3$ ,  $c_{m_0+1} = -\varepsilon + (2m_0 + 1)Q/3$ ,  $c_{m_0+2} = \varepsilon + C - m_0Q/3$ . Namely if  $j \leq 0$ , then

$$\begin{aligned}(\eta_{m_0-2N+3j}, Y_{m_0-2N+3j}) &= (+1, 2jQ + \varepsilon), \\ (\zeta_{m_0-2N+3j}, Z_{m_0-2N+3j}) &= (\chi, jQ - C), \\ (\eta_{m_0-2N+3j+1}, Y_{m_0-2N+3j+1}) &= (\chi, (m_0 + 2j + 1)Q - C - \varepsilon), \\ (\zeta_{m_0-2N+3j+1}, Z_{m_0-2N+3j+1}) &= (+1, (m_0 + j + 1)Q - \varepsilon), \\ (\eta_{m_0-2N+3j+2}, Y_{m_0-2N+3j+2}) &= (\chi, (m_0 + 2j + 2)Q + C), \\ (\zeta_{m_0-2N+3j+2}, Z_{m_0-2N+3j+2}) &= (\chi, (j + 1)Q + C + \varepsilon),\end{aligned}\tag{90}$$

whose expression is slightly different from Eq.(89). Recall that the Airy-type solution for  $m \leq m_0 - 2N - 1$  is written as Eq.(43), i.e.

$$(\zeta^{(N)}(m), Z^{(N)}(m)) = (+1, (-m - 2N - 1)Q),\tag{91}$$

and completely different from the solution in Eq.(89) or Eq.(90), whose initial value is perturbed.

We now investigate the evolution for  $m \geq m_0 + N + 1$ . Recall that

$$\begin{aligned}(\eta_{m_0+N+1}, Y_{m_0+N+1}) &= ((-1)^N \chi, (m_0 - N^2 + N + 1)Q - C - (N + 1)\varepsilon), \\(\zeta_{m_0+N+1}, Z_{m_0+N+1}) &= (+1, (m_0 + N + 1)Q - \varepsilon).\end{aligned}\tag{92}$$

If  $\varepsilon > 0$ , then  $Z_{m_0+N+1} < (m_0 + N + 1)Q$  and we have

$$\begin{aligned}(\eta_{m_0+N+2}, Y_{m_0+N+2}) &= ((-1)^N \chi, (m_0 + N^2 + 3N + 2)Q + C + N\varepsilon), \\(\zeta_{m_0+N+2}, Z_{m_0+N+2}) &= \begin{cases} ((-1)^N \chi, (N + 1)^2 Q + C + (N + 1)\varepsilon), & (Y_{m_0+N+2} > 0) \\ (-1, -(m_0 + N + 1)Q + \varepsilon), & (Y_{m_0+N+2} < 0). \end{cases}\end{aligned}\tag{93}$$

If  $\varepsilon < 0$ , then  $Z_{m_0+N+1} > (m_0 + N + 1)Q$  and we have

$$\begin{aligned}(\eta_{m_0+N+2}, Y_{m_0+N+2}) &= ((-1)^{N+1} \chi, (m_0 + N^2 + 3N + 2)Q + C + (N + 1)\varepsilon), \\(\zeta_{m_0+N+2}, Z_{m_0+N+2}) &= \begin{cases} ((-1)^{N+1} \chi, (N + 1)^2 Q + C + (N + 2)\varepsilon), & (Y_{m_0+N+2} > 0) \\ (-1, -(m_0 + N + 1)Q + \varepsilon), & (Y_{m_0+N+2} < 0). \end{cases}\end{aligned}\tag{94}$$

Therefore they are completely different from the Airy-type solution for  $m_0 + N + 2 \leq m \leq -1$  written as

$$(\zeta^{(N)}(m), Z^{(N)}(m)) = (+1, mQ).\tag{95}$$

We discuss the case  $k_0 \neq 0$  with the condition  $-m_0 Q - (N - k_0)(N - k_0 + 1)Q < C < -m_0 Q - (N - k_0 + 1)(N - k_0 + 2)Q (< (m_0 + 1)Q)$ . Then it is shown that the function in Eq.(85) satisfies the condition in Eq.(86) for  $m_0 - 2N \leq m \leq m_0 + N - 3k_0 + 1$ ,  $0 \leq j \leq N - k_0$  and  $0 \leq j' \leq N - k_0 - 1$ . Hence the function  $(\eta_m, Y_m)$  for  $m_0 - 2N \leq m \leq m_0 + N - 3k_0 + 2$  and  $(\zeta_m, Z_m)$  for  $m_0 - 2N \leq m \leq m_0 + N - 3k_0 + 1$  in Eq.(85) satisfies p-ud PII. However we have

$$\begin{aligned}(\eta_{m_0+N-3k_0+2}, Y_{m_0+N-3k_0+2}) &= \\((-1)^{N-k_0} \chi, (m_0 + (N - k_0 + 1)(N - k_0 + 2))Q + C + (N - k_0)\varepsilon),\end{aligned}\tag{96}$$

and  $Y_{m_0+N-3k_0+2} < 0$ , namely the condition is changed. By applying the evolution of p-ud PII, we have

$$\begin{aligned}(\zeta_{m_0+N-3k_0+2}, Z_{m_0+N-3k_0+2}) &= (-1, -(m_0 + N - k_0 + 1)Q + \varepsilon), \\(\eta_{m_0+N-3k_0+3}, Y_{m_0+N-3k_0+3}) &= \\((-1)^{N-k_0+1} \chi, -(m_0 + (N - k_0 + 1)(N - k_0))Q - C - (N - k_0 - 1)\varepsilon),\end{aligned}\tag{97}$$

and  $Y_{m_0+N-3k_0+3} < 0$ . Then

$$(\zeta_{m_0+N-3k_0+2}, Z_{m_0+N-3k_0+2}) = (+1, (m_0 + N - k_0 + 1)Q - \varepsilon).\tag{98}$$

To obtain further evolution, we set  $D' = \varepsilon$ ,  $A = (2N+1)Q$ ,  $m' = m_0 - 2N - 1$ ,  $k = j$ ,  $C' = 2m_0Q + C - 3\varepsilon$ ,  $\eta = +1$ ,  $\zeta = \chi$  in Eq.(78). Then we have

$$\begin{aligned}
(\eta_{m_0-2N+3j}, Y_{m_0-2N+3j}) &= ((-1)^j \chi, (-m_0 - j(j-1))Q - C - (j-2)\varepsilon), \\
(\zeta_{m_0-2N+3j}, Z_{m_0-2N+3j}) &= (+1, (m_0 + j)Q - \varepsilon), \\
(\eta_{m_0-2N+3j+1}, Y_{m_0-2N+3j+1}) &= ((-1)^j \chi, (3m_0 + j^2 + 3j + 1)Q + C + (j-3)\varepsilon), \\
(\zeta_{m_0-2N+3j+1}, Z_{m_0-2N+3j+1}) &= (-1)^j \chi, (2m_0 + (j+1)^2)Q + C + (j-2)\varepsilon, \\
(\eta_{m_0-2N+3j+2}, Y_{m_0-2N+3j+2}) &= (+1, (2j+2)Q + \varepsilon), \\
(\zeta_{m_0-2N+3j+2}, Z_{m_0-2N+3j+2}) &= ((-1)^j \chi, -(2m_0 + (j+1)^2)Q - C - (j-2)\varepsilon),
\end{aligned} \tag{99}$$

and the condition

$$Z_m < mQ, Y_{m_0-2N+3j+1} > 0, Y_{m_0-2N+3j+2} < 0, Y_{m_0-2N+3j+3} > 0 \tag{100}$$

is satisfied for  $m_0 + N - 3k_0 + 3 \leq m \leq m_0 + N - 1$ ,  $N - k_0 + 1 \leq j \leq N - 1$ . Hence the functions  $(\eta_m, Y_m)$  for  $m_0 + N - 3k_0 + 3 \leq m \leq m_0 + N$  and  $(\zeta_m, Z_m)$  for  $m_0 + N - 3k_0 + 3 \leq m \leq m_0 + N$  in Eq.(99) satisfy p-ud PII. If  $\varepsilon = 0$ , then the function defined by Eqs.(85), (97), (99) coincides with the solution in section 3, which has indefinite evolution caused by  $(\eta_{m_0-2N}, Y_{m_0-2N}) = (+1, 0)$  and  $(\zeta_{m_0+N}, Z_{m_0+N}) = (+1, (m_0 + N)Q)$ . By imposing  $\varepsilon \neq 0$ , the indefiniteness of the evolution may disappear. The solutions for  $m \leq m_0 - 2N - 1$  is obtained exactly the same as Eqs.(89), (90), and they are different from the Airy-type solutions. The solutions for  $m_0 + N + 1 \leq m$  are also different from the Airy-type solutions, although we need case classification to express the solutions.

We give an example which is related with the solution in Eqs.(4), (14). Set  $Q = -3$  and  $N = 3$  ( $A = 7Q$ ) and choose the initial value as  $(\eta_{-18}, Y_{-18}) = (+1, 0 - \varepsilon)$ ,  $(\zeta_{-18}, Z_{-18}) = (+1, 29)$  ( $0 < 4\varepsilon < 1$ ). Then the solution of p-ud PII

is written as follows:

$$(\eta_m, Y_m), (\zeta_m, Z_m) = \left\{ \begin{array}{lll} (+1, 6 - \varepsilon), & (-1, 32 - \varepsilon) & (m = -21) \\ (-1, 68), & (+1, 36 + \varepsilon) & (m = -20) \\ (-1, 7 + \varepsilon), & (-1, -29) & (m = -19) \\ (+1, 0 - \varepsilon), & (+1, 29) & (m = -18) \\ (+1, 62 + 2\varepsilon), & (+1, 33 + \varepsilon) & (m = -17) \\ (+1, 1), & (+1, -32 - \varepsilon) & (m = -16) \\ (+1, -6 - \varepsilon), & (-1, 32 + \varepsilon) & (m = -15) \\ (-1, 62 - 2\varepsilon), & (+1, 30 + \varepsilon) & (m = -14) \\ (-1, -11 - \varepsilon), & (-1, -30 - \varepsilon) & (m = -13) \\ (+1, -1), & (+1, 30 + \varepsilon) & (m = -12) \\ (+1, 46 + \varepsilon), & (+1, 16) & (m = -11) \\ (+1, -18 - \varepsilon), & (-1, -16) & (m = -10) \\ (-1, 11 + \varepsilon), & (+1, 27 + \varepsilon) & (m = -9) \\ (+1, 22 - \varepsilon), & (+1, -5 - 2\varepsilon) & (m = -8) \\ (+1, -24 - \varepsilon), & (-1, 5 + 2\varepsilon) & (m = -7) \\ (-1, 29 + 3\varepsilon), & (+1, 24 + \varepsilon) & (m = -6) \\ (+1, -14 - 3\varepsilon), & (-1, -24 - \varepsilon) & (m = -5) \\ (-1, -16 + 2\varepsilon), & (+1, 24 + \varepsilon) & (m = -4) \\ (+1, 19 - 2\varepsilon), & (+1, -5 - 3\varepsilon) & (m = -3) \\ (+1, -36 - \varepsilon), & (-1, 5 + 3\varepsilon) & (m = -2) \\ (-1, 26 + 4\varepsilon), & (+1, 21 + \varepsilon) & (m = -1) \\ (+1, -41 - 4\varepsilon), & (-1, -21 - \varepsilon) & (m = 0) \\ (-1, -1 + 3\varepsilon), & (+1, 21 + \varepsilon) & (m = 1) \\ (+1, -26 - 3\varepsilon), & (-1, -21 - \varepsilon) & (m = 2) \\ (-1, -21 + 2\varepsilon), & (+1, 21 + \varepsilon) & (m = 3) \\ (+1, -17 - 2\varepsilon), & (-1, -21 - \varepsilon) & (m = 4) \\ (-1, -37 + \varepsilon), & (+1, 21 + \varepsilon) & (m = 5) \\ (+1, -14 - \varepsilon), & (-1, -21 - \varepsilon) & (m = 6) \\ (-1, -46), & (+1, 21 + \varepsilon) & (m = 7) \\ (+1, -17), & (-1, -21 - \varepsilon) & (m = 8). \end{array} \right. \quad (101)$$

If we impose  $\varepsilon = 0$ , then the function coincides with the Airy-type solution given in Eqs.(4), (14) for  $-18 \leq m \leq -9$ . The values  $(\eta_{-18}, Y_{-18}) = (+1, 0 - \varepsilon)$  and  $(\zeta_{-9}, Z_{-9}) = (+1, 27 + \varepsilon)$  in the case  $\varepsilon = 0$  show indefinite evolution, although the evolution is unique in the case  $\varepsilon \neq 0$ . Assume that  $0 < \varepsilon < 1/4$ . In the case  $m \leq -19$ , the solution is written in the form of Proposition 11, namely

$$(\eta_m, Y_m), (\zeta_m, Z_m) = \left\{ \begin{array}{lll} (+1, 6j - 36 - \varepsilon), & (-1, 3j + 11 - \varepsilon) & (m = -3j) \\ (-1, 6j + 26), & (+1, 3j + 15 + \varepsilon) & (m = -3j + 1) \\ (-1, 6j - 35 + \varepsilon), & (-1, 3j - 50) & (m = -3j + 2) \end{array} \right. \quad (102)$$

for  $j \geq 7$ , and it is quite different from the Airy-type solution written as  $(\zeta_m, Z_m) = (+1, 3m + 21)$  in Eq.(4). If  $-18 \leq m \leq -9$ , then the solution is written in the form of Eqs.(85), (99). In the case  $0 \leq m \leq 6$ , the solution



is written in the form of Eqs.(81), (82). In the case  $m \geq 7$ , the solution is written in the form of Proposition 10, namely

$$(\eta_m, Y_m), (\zeta_m, Z_m) = \begin{cases} (-1, -6j - 22), & (+1, 21 + \varepsilon) & (m = 2j - 1) \\ (+1, -6j + 7), & (-1, -21 - \varepsilon) & (m = 2j) \end{cases} \quad (103)$$

for  $j \geq 4$ , and it is quite different from the Airy-type solution in Eq.(4).

If the initial value  $Y_{-18}$  is positive, then the solution is written in a slightly different form. We choose the initial value as  $(\eta_{-18}, Y_{-18}) = (+1, 0 + \varepsilon)$ ,  $(\zeta_{-18}, Z_{-18}) = (+1, 29)$  ( $0 < 3\varepsilon < 1$ ). Then the solution of p-ud PII is written as follows:

$$(\eta_m, Y_m), (\zeta_m, Z_m) = \left\{ \begin{array}{lll} (+1, 6j - 36 + \varepsilon), & (+1, 3j + 11) & (m = -3j, j \geq 6) \\ (+1, 6j + 26 - \varepsilon), & (+1, 3j + 15 - \varepsilon) & (m = -3j + 1, j \geq 6) \\ (+1, 6j - 35), & (+1, 3j - 50 + \varepsilon) & (m = -3j + 2, j \geq 6) \\ (+1, -6 + \varepsilon), & (-1, 32 - \varepsilon) & (m = -15) \\ (-1, 62 - 2\varepsilon), & (+1, 30 - \varepsilon) & (m = -14) \\ (-1, -11 + \varepsilon), & (-1, -30 + \varepsilon) & (m = -13) \\ (+1, -1), & (+1, 30 - \varepsilon) & (m = -12) \\ (+1, 46 - \varepsilon), & (+1, 16) & (m = -11) \\ (+1, -18 + \varepsilon), & (-1, -16) & (m = -10) \\ (-1, 11 - \varepsilon), & (-1, 27 - \varepsilon) & (m = -9) \\ (-1, 22), & (+1, -5 + \varepsilon) & (m = -8) \\ (+1, -24 + \varepsilon), & (-1, 5 - \varepsilon) & (m = -7) \\ (+1, 29 - 2\varepsilon), & (+1, 24 - \varepsilon) & (m = -6) \\ (-1, -14 + 2\varepsilon), & (-1, -24 + \varepsilon) & (m = -5) \\ (+1, -16 - \varepsilon), & (+1, 24 - \varepsilon) & (m = -4) \\ (-1, 19 + \varepsilon), & (-1, -5 + 2\varepsilon) & (m = -3) \\ (+1, -36 + \varepsilon), & (+1, 5 - 2\varepsilon) & (m = -2) \\ (+1, 26 - 3\varepsilon), & (+1, 21 - \varepsilon) & (m = -1) \\ (-1, -41 + 3\varepsilon), & (-1, -21 + \varepsilon) & (m = 0) \\ (+1, -1 - 2\varepsilon), & (+1, 21 - \varepsilon) & (m = 1) \\ (-1, -26 + 2\varepsilon), & (-1, -21 + \varepsilon) & (m = 2) \\ (+1, -21 - \varepsilon), & (+1, 21 - \varepsilon) & (m = 3) \\ (-1, -17 + \varepsilon), & (-1, -21 + \varepsilon) & (m = 4) \\ (+1, -37), & (+1, 21 - \varepsilon) & (m = 5) \\ (-1, -14), & (-1, -21 + \varepsilon) & (m = 6) \\ (+1, -6j - 22 + \varepsilon), & (+1, 21 - \varepsilon) & (m = 2j - 1, j \geq 4) \\ (-1, -6j + 7 - \varepsilon), & (-1, -21 + \varepsilon) & (m = 2j, j \geq 4). \end{array} \right. \quad (104)$$

Let us investigate the solutions of p-ud PII which are close to the function  $(\eta^{(M)}(m), Y^{(M)}(m)), (\zeta^{(M)}(m), Z^{(M)}(m))$  in section 3. We assume that the constant  $A$  in the p-ud-PII is written as  $A = (2M + 1)Q$ ,  $M \in \mathbb{Z}_{\leq -1}$ ,  $k_0 \in \{0, 1, \dots, N\}$ ,  $\chi \in \{+1, -1\}$  and the value  $m_0 \in \mathbb{Z}_{< 0}$  satisfies Eqs.(53). Let  $\varepsilon$  be a real number such that  $|\varepsilon|$  is sufficiently small.

We set  $A = (2M + 1)Q$ ,  $m' = m_0 + M + 1$ ,  $k = j - 1$ ,  $C' = (M + 1)Q + C$ ,  $D' = \varepsilon$ ,  $\eta = -\chi$ ,  $\zeta = +1$  in Eq.(80). Then we have

$$\begin{aligned}
(\zeta_{m_0+M+3j-2}, Z_{m_0+M+3j-2}) &= (+1, (m_0 + M + j)Q + \varepsilon), \\
(\eta_{m_0+M+3j-1}, Y_{m_0+M+3j-1}) &= ((-1)^{j-1}\chi, ((m_0 + 2M + j(j + 1))Q - (j - 1)\varepsilon + C), \\
(\zeta_{m_0+M+3j-1}, Z_{m_0+M+3j-1}) &= ((-1)^{j-1}\chi, (j^2 + M)Q - j\varepsilon + C), \\
(\eta_{m_0+M+3j}, Y_{m_0+M+3j}) &= (+1, 2(j + M)Q - \varepsilon), \\
(\zeta_{m_0+M+3j}, Z_{m_0+M+3j}) &= ((-1)^{j-1}\chi, (-j^2 + 2j + M)Q + (j - 1)\varepsilon - C), \\
(\eta_{m_0+M+3j+1}, Y_{m_0+M+3j+1}) &= ((-1)^{j-1}\chi, (m_0 + 2M - j^2 + 3j + 1)Q + j\varepsilon - C),
\end{aligned} \tag{105}$$

We discuss the case  $k_0 = 0$  with the condition  $(-m_0 - (M + 1)M)Q < C < (m_0 + 1)Q$ . Then the function in Eq.(105) satisfies the conditions

$$\begin{aligned}
Y_m &> 0, \quad Z_{m_0+M+3j-1} < (m_0 + M + 3j - 1)Q, \\
Z_{m_0+M+3j'} &< (m_0 + M + 3j')Q, \quad Z_{m_0+M+3j'+1} > (m_0 + M + 3j' + 1)Q
\end{aligned} \tag{106}$$

for  $m_0 + M + 2 \leq m \leq m_0 - 2M - 1$ ,  $1 \leq j \leq -M$  and  $1 \leq j' \leq -M - 1$ . Hence the function  $(\eta_m, Y_m)$  for  $m_0 + M + 2 \leq m \leq m_0 - 2M$  and  $(\zeta_m, Z_m)$  for  $m_0 + M + 1 \leq m \leq m_0 - 2M - 1$  in Eq.(105) satisfies p-ud PII. If  $\varepsilon = 0$ , then the function defined by Eq.(105) coincides with the solution in Propositions 7 (ii) and 8 (i), which has indefinite evolution caused by  $(\zeta_{m_0+M+1}, Z_{m_0+M+1}) = (+1, (m_0 + M + 1)Q)$  and  $(\eta_{m_0-2M}, Y_{m_0-2M}) = (+1, 0)$ . By imposing  $\varepsilon \neq 0$ , the indefiniteness of the evolution may disappear. We investigate the evolution for  $m \leq m_0 + M + 1$ . Recall that

$$\begin{aligned}
Z_{m_0+M+1} &= (m_0 + M + 1)Q + \varepsilon, \quad \zeta = +1, \\
Y_{m_0+M+2} &= (m_0 + 2M + 2)Q + C, \quad \eta = \chi.
\end{aligned} \tag{107}$$

If  $\varepsilon > 0$ , then  $Z_{m_0+M+1} > (m_0 + M + 1)Q$  and we have

$$\begin{aligned}
(\eta_{m_0+M+1}, Y_{m_0+M+1}) &= (-\chi, (m_0 + 2M - 1)Q - C), \\
(\zeta_{m_0+M}, Z_{m_0+M}) &= (-\chi, MQ - \varepsilon - C), \\
(\eta_{m_0+M}, Y_{m_0+M}) &= (+1, 2MQ - \varepsilon), \\
(\zeta_{m_0+M-1}, Z_{m_0+M-1}) &= (-\chi, MQ + C), \\
(\eta_{m_0+M-1}, Y_{m_0+M-1}) &= (-\chi, (m_0 + 2M)Q + \varepsilon + C), \\
(\zeta_{m_0+M-2}, Z_{m_0+M-2}) &= (+1, (m_0 + M)Q + \varepsilon),
\end{aligned} \tag{108}$$

and the conditions  $Z_m < mQ$  for  $m = m_0 + M, m_0 + M - 1, m_0 + M - 2$  and  $Y_m > 0$  for  $m = m_0 + M + 1, m_0 + M + 2, m_0 + M + 3$  are satisfied. Then we can use Proposition 11 (ii), and the solution is expressed in the form of Proposition 11 (i) by setting  $c_{m_0} = -C - \varepsilon - (m_0 + 1)Q/3$ ,  $c_{m_0+1} = \varepsilon + (2m_0 + 1)Q/3$ ,

$c_{m_0+2} = C - m_0Q/3$ . Namely if  $j < 0$ , then

$$\begin{aligned}
(\eta_{m_0+M+3j-2}, Y_{m_0+M+3j-2}) &= (-\chi, (m_0 + 2M + 2j - 1)Q - C), \\
(\zeta_{m_0+M+3j-2}, Z_{m_0+M+3j-2}) &= (+1, (m_0 + M + j)Q + \varepsilon), \\
(\eta_{m_0+M+3j-1}, Y_{m_0+M+3j-1}) &= (-\chi, (m_0 + 2M + 2j)Q + \varepsilon + C), \\
(\zeta_{m_0+M+3j-1}, Z_{m_0+M+3j-1}) &= (-\chi, (M + j)Q + C), \\
(\eta_{m_0+M+3j}, Y_{m_0+M+3j}) &= (+1, 2(M + j)Q - \varepsilon), \\
(\zeta_{m_0+M+3j}, Z_{m_0+M+3j}) &= (-\chi, (M + j)Q - \varepsilon - C),
\end{aligned} \tag{109}$$

If  $\varepsilon < 0$ , then we have  $Z_{m_0+M+1} < (m_0 + M + 1)Q$  and the conditions  $Z_m < mQ$  for  $m = m_0 + M + 3, m_0 + M + 2, m_0 + M + 1$  and  $Y_m > 0$  for  $m = m_0 + M + 4, m_0 + M + 3, m_0 + M + 2$  are satisfied. Then we can use Proposition 11 (ii) and the solution is expressed in the form of Proposition 11 (i) by setting  $c_{m_0} = -C - (m_0 + 1)Q/3$ ,  $c_{m_0+1} = \varepsilon + (2m_0 + 1)Q/3$ ,  $c_{m_0+2} = C - \varepsilon - m_0Q/3$ , i.e. we have

$$\begin{aligned}
(\eta_{m_0+M+3j-2}, Y_{m_0+M+3j-2}) &= (\chi, (m_0 + 2M + 2j - 1)Q + \varepsilon - C), \\
(\zeta_{m_0+M+3j-2}, Z_{m_0+M+3j-2}) &= (+1, (m_0 + M + j)Q + \varepsilon), \\
(\eta_{m_0+M+3j-1}, Y_{m_0+M+3j-1}) &= (\chi, (m_0 + 2M + 2j)Q + C), \\
(\zeta_{m_0+M+3j-1}, Z_{m_0+M+3j-1}) &= (\chi, (M + j)Q - \varepsilon + C), \\
(\eta_{m_0+M+3j}, Y_{m_0+M+3j}) &= (+1, 2(M + j)Q - \varepsilon), \\
(\zeta_{m_0+M+3j}, Z_{m_0+M+3j}) &= (\chi, (M + j)Q - C),
\end{aligned} \tag{110}$$

for  $j \leq 0$ . Recall that the Airy-type solution for  $m \leq m_0 + M$  is written as Eq.(55) and completely different from the solution in Eq.(109) or Eq.(110). We now investigate the evolution for  $m \geq m_0 - 2M$ . By recalling the values  $(\zeta_{m_0-2M-1}, Z_{m_0-2M-1})$  and  $(\eta_{m_0-2M}, Y_{m_0-2M})$ , we have

$$(\zeta_{m_0-2M}, Z_{m_0-2M}) = \begin{cases} ((-1)^M \chi, -(M^2 + M)Q + M\varepsilon - C), & \varepsilon > 0, \\ ((-1)^{M-1} \chi, -(M^2 + M)Q + (M - 1)\varepsilon - C), & \varepsilon < 0. \end{cases} \tag{111}$$

Therefore it is completely different from the Airy-type solution for  $m_0 - 2M \leq m \leq M + 1$  written as  $(\zeta^{(M)}(m), Z^{(M)}(m)) = (+1, (-m - 2M - 1)Q)$ .

Let us consider the case  $k_0 \neq 0$  with the condition  $(-m_0 - (M + k_0 + 1)(M + k_0))Q \leq C < (-m_0 - (M + k_0)(M + k_0 - 1))Q$ . The condition in Eq.(106) is satisfied for  $m_0 + M + 2 \leq m \leq m_0 - 2M - 3k_0$ ,  $1 \leq j \leq -M - k_0$  and  $1 \leq j' \leq -M - k_0 - 1$ . Hence the function  $(\eta_m, Y_m)$  for  $m_0 + M + 2 \leq m \leq m_0 - 2M - 3k_0$  and  $(\zeta_m, Z_m)$  for  $m_0 + M + 1 \leq m \leq m_0 - 2M - 3k_0$  in Eq.(105) satisfies p-ud PII. We have  $Z_{m_0+M+3j} - (m_0 + M + 3j)Q = -C - m_0Q + (M + k_0)(M + k_0 - 1)Q + (j - 1)\varepsilon > 0$ , and the condition is changed. By applying evolution of p-ud

PII, we have

$$\begin{aligned}
(\eta_{m_0-2M-3k_0+1}, Y_{m_0-2M-3k_0+1}) &= (-1, (2m_0 - 2M - 4k_0 + 1)Q + \varepsilon), \quad (112) \\
(\zeta_{m_0-2M-3k_0+1}, Z_{m_0-2M-3k_0+1}) &= \\
&((-1)^{-M-k_0}, (2m_0 + M + (-M - k_0 + 1)^2)Q + (M + k_0 + 2)\varepsilon + C), \\
(\eta_{m_0-2M-3k_0+2}, Y_{m_0-2M-3k_0+2}) &= (+1, (-2k_0 + 2)Q - \varepsilon).
\end{aligned}$$

Note that  $Z_{m_0-2M-3k_0+1} > (m_0 - 2M - 3k_0 + 1)Q$ . To obtain further evolution, we set  $D' = \varepsilon$ ,  $A = (2M + 1)Q$ ,  $m' = m_0 + M$ ,  $k = j - 1$ ,  $C' = (2m_0 + M + 1)Q + C + 3\varepsilon$ ,  $\eta = -\chi$ ,  $\zeta = +1$  in Eq.(80). Then we have

$$\begin{aligned}
(\zeta_{m_0+M+3j-2}, Z_{m_0+M+3j-2}) &= ((-1)^{j-1}\chi, (2m_0 + M + j^2)Q - (j - 3)\varepsilon + C), \\
(\eta_{m_0+M+3j-1}, Y_{m_0+M+3j-1}) &= (+1, 2(M + j)Q - \varepsilon), \quad (113) \\
(\zeta_{m_0+M+3j-1}, Z_{m_0+M+3j-1}) &= ((-1)^{j-1}\chi, (-2m_0 + M - j^2 + 2j)Q + (j - 4)\varepsilon - C), \\
(\eta_{m_0+M+3j}, Y_{m_0+M+3j}) &= ((-1)^{j-1}, (-m_0 + 2M - j^2 + 3j)Q + (j - 3)\varepsilon - C), \\
(\zeta_{m_0+M+3j}, Z_{m_0+M+3j}) &= (+1, (m_0 + M + j)Q + \varepsilon), \\
(\eta_{m_0+M+3j+1}, Y_{m_0+M+3j+1}) &= ((-1)^{j-1}\chi, (3m_0 + 2M + j^2 + 3j + 1)Q - (j - 3)\varepsilon + C).
\end{aligned}$$

The conditions  $Y_m > 0$ ,  $Z_{m_0+M+3j-1} < (m_0 + M + 3j - 1)Q$ ,  $Z_{m_0+M+3j} > (m_0 + M + 3j)Q$ ,  $Z_{m_0+M+3j+1} < (m_0 + M + 3j + 1)Q$  are satisfied for  $m_0 - 2M - 3k_0 + 2 \leq m \leq m_0 - 2M$ ,  $-M - k_0 + 1 \leq j \leq -M - 1$ . Hence the functions  $(\eta_m, Y_m)$  for  $m_0 - 2M - 3k_0 + 2 \leq m \leq m_0 - 2M - 1$  and  $(\zeta_m, Z_m)$  for  $m_0 - 2M - 3k_0 + 1 \leq m \leq m_0 - 2M - 2$  in Eq.(113) satisfy p-ud PII. Note that, if  $\varepsilon = 0$ , then the function defined by Eqs.(105), (112), (113) coincides with the solution  $(\eta^{(M)}(m), Y^{(M)}(m))$ ,  $(\zeta^{(M)}(m), Z^{(M)}(m))$  in section 3, which has indefinite evolution caused by  $(\zeta_{m_0+M+1}, Z_{m_0+M+1}) = (+1, (m_0 + M + 1)Q)$  and  $(\eta_{m_0-2M-1}, Y_{m_0-2M-1}) = (+1, 0)$ . By imposing  $\varepsilon \neq 0$ , the indefiniteness of the evolution may disappear. The solutions for  $m \leq m_0 + M + 1$  are described exactly the same as Eqs.(109), (110), and they are different from the Airy-type solutions. The solutions for  $m_0 - 2M \leq m$  are obtained similarly to the case  $k_0 = 0$ , and they are also different from the Airy-type solutions.

We give an example which is related with the solution in Eqs.(64), (65). Set  $Q = -2$  and  $M = -3$  ( $A = -5Q$ ) and choose the initial value as  $(\zeta_{-12}, Z_{-12}) = (+1, 24 + \varepsilon)$ ,  $(\eta_{-11}, Y_{-11}) = (+1, 18)$  ( $0 < 3\varepsilon < 1$ ). Then the solution of p-ud

PII is written as follows:

$$(\eta_m, Y_m), (\zeta_m, Z_m) = \begin{cases} (-1, 22 + \varepsilon), & (-1, -4) & (m = -14) \\ (+1, 12 - \varepsilon), & (-1, 16 - \varepsilon) & (m = -13) \\ (-1, 40), & (+1, 24 + \varepsilon) & (m = -12) \\ (+1, 18), & (+1, -6 - \varepsilon) & (m = -11) \\ (+1, 8 - \varepsilon), & (+1, 14) & (m = -10) \\ (+1, 36 + \varepsilon), & (+1, 22 + \varepsilon) & (m = -9) \\ (-1, 10 - \varepsilon), & (-1, -12 - 2\varepsilon) & (m = -8) \\ (+1, 4 - \varepsilon), & (-1, 16 + \varepsilon) & (m = -7) \\ (-1, 34 + \varepsilon), & (+1, 18) & (m = -6) \\ (+1, 0 - \varepsilon), & (-1, -18) & (m = -5) \\ (-1, 2 + \varepsilon), & (+1, 20 + \varepsilon) & (m = -4) \\ (+1, 24 - \varepsilon), & (+1, 4 - 2\varepsilon) & (m = -3) \\ (+1, -4 - \varepsilon), & (-1, -4 + 2\varepsilon) & (m = -2) \\ (-1, 14 + 3\varepsilon), & (+1, 18 + \varepsilon) & (m = -1) \\ (+1, 0 - 3\varepsilon), & (+1, -18 - \varepsilon) & (m = 0) \\ (+1, -8 + 2\varepsilon), & (-1, 18 + \varepsilon) & (m = 1) \\ (-1, 14 - 2\varepsilon), & (+1, -4 - 3\varepsilon) & (m = 2) \\ (+1, -12 - \varepsilon), & (-1, 4 + 3\varepsilon) & (m = 3) \\ (-1, 10 + \varepsilon), & (+1, 6 - 2\varepsilon) & (m = 4) \\ (+1, -16 - \varepsilon), & (-1, -6 + 2\varepsilon) & (m = 5) \\ (-1, 6 + \varepsilon), & (+1, 12 - \varepsilon) & (m = 6) \\ (+1, -20 - \varepsilon), & (-1, -12 + \varepsilon) & (m = 7) \\ (-1, 2 + \varepsilon), & (+1, 14) & (m = 8) \\ (+1, -24 - \varepsilon), & (-1, -14) & (m = 9) \\ (-1, -2 + \varepsilon), & (+1, 14) & (m = 10). \end{cases} \quad (114)$$

The function  $(\eta_m, Y_m)$  (resp.  $(\zeta_m, Z_m)$ ) for  $-11 \leq m \leq -5$  (resp.  $-12 \leq m \leq -6$ ) is written in the form of Eqs.(83), (84), and they coincide with the Airy-type solution given in Eq.(65) (resp. Eq.(64)) with the additional condition  $\varepsilon = 0$ . The values  $(\zeta_{-12}, Z_{-12}) = (+1, 24 + \varepsilon)$  and  $(\eta_{-5}, Y_{-5}) = (+1, 0 - \varepsilon)$  for the case  $\varepsilon = 0$  show indefinite evolution, although the evolution is unique in the case  $\varepsilon \neq 0$ . Assume that  $\varepsilon \neq 0$ . In the case  $m \geq -12$ , the solution is written in the form of Proposition 11, namely

$$(\eta_m, Y_m), (\zeta_m, Z_m) = \begin{cases} (-1, 4j + 24), & (+1, 2j + 16 + \varepsilon) & (m = -3j, j \geq 4) \\ (-1, 4j + 2 + \varepsilon), & (-1, 2j - 14) & (m = -3j + 1, j \geq 5) \\ (+1, 4j - 8 - \varepsilon), & (-1, 2j + 6 - \varepsilon) & (m = -3j + 2, j \geq 5) \end{cases} \quad (115)$$

and it is quite different from the Airy-type solution written as  $(\zeta_m, Z_m) = (+1, -2m)$  in Eq.(64). In the case  $3 \leq m \leq 9$ , the solution is written in the form of Eqs.(83), (84). In the case  $m \geq 9$ , the solution is written in the form

of Proposition 10, namely

$$(\eta_m, Y_m), (\zeta_m, Z_m) = \begin{cases} (+1, -4j - 4 - \varepsilon), & (-1, -14) & (m = 2j - 1, j \geq 5) \\ (-1, -4j + 18 + \varepsilon), & (+1, 14) & (m = 2j, j \geq 5) \end{cases} \quad (116)$$

and it is quite different from the Airy-type solution written as  $(\zeta_m, Z_m) = (+1, (-m - 2M - 1)Q)$ .

If the initial value satisfies  $Z_{-12} > 24$ , then the solution is written in a slightly different form. We choose the initial value as  $(\zeta_{-12}, Z_{-12}) = (+1, 24 + \varepsilon)$ ,  $(\eta_{-11}, Y_{-11}) = (+1, 18)$  ( $0 < 3\varepsilon < 1$ ). Then the solution of p-ud PII is written as follows:

$$(\eta_m, Y_m), (\zeta_m, Z_m) = \left\{ \begin{array}{lll} (-1, 4j + 24 - \varepsilon), & (+1, 2j + 16 - \varepsilon) & (m = -3j, j \geq 3) \\ (-1, 4j + 2), & (-1, 2j - 14 + \varepsilon) & (m = -3j + 1, j \geq 4) \\ (+1, 4j - 8 + \varepsilon), & (-1, 2j + 6) & (m = -3j + 2, j \geq 4) \\ (-1, 10 + \varepsilon), & (-1, -12 + 2\varepsilon) & (m = -8) \\ (+1, 4 + \varepsilon), & (-1, 16 - \varepsilon) & (m = -7) \\ (-1, 34 - \varepsilon), & (+1, 18) & (m = -6) \\ (+1, 0 + \varepsilon), & (+1, -18 + \varepsilon) & (m = -5) \\ (+1, 2), & (+1, 20 - \varepsilon) & (m = -4) \\ (-1, 24), & (-1, 4 + \varepsilon) & (m = -3) \\ (+1, -4 + \varepsilon), & (+1, -4 - \varepsilon) & (m = -2) \\ (+1, 14 - 2\varepsilon), & (+1, 18 - \varepsilon) & (m = -1) \\ (-1, 0 + 2\varepsilon), & (-1, -18 + 3\varepsilon) & (m = 0) \\ (+1, -8 + \varepsilon), & (+1, 18 - 3\varepsilon) & (m = 1) \\ (-1, 14 - \varepsilon), & (-1, -4 + 2\varepsilon) & (m = 2) \\ (+1, -12 + \varepsilon), & (+1, 4 - 2\varepsilon) & (m = 3) \\ (-1, 10 - \varepsilon), & (-1, 6 + \varepsilon) & (m = 4) \\ (+1, -16 + \varepsilon), & (+1, -6 - \varepsilon) & (m = 5) \\ (-1, 6 - \varepsilon), & (-1, 12) & (m = 6) \\ (+1, -20 + \varepsilon), & (+1, -12) & (m = 7) \\ (-1, 2 - \varepsilon), & (-1, 14 - \varepsilon) & (m = 8) \\ (+1, -4j - 4 + \varepsilon), & (+1, -14 + \varepsilon) & (m = 2j - 1, j \geq 5) \\ (-1, -4j + 18 - \varepsilon), & (-1, 14 - \varepsilon) & (m = 2j, j \geq 5). \end{array} \right. \quad (117)$$

## 6 Concluding Remarks

In this paper we introduced the simultaneous p-ud PII by setting  $y(q^{m+1}) = z(q^{m+1})z(q^m) + 1$  in Eq.(1) and discussed its solutions. In particular we investigated the solutions of p-ud PII which are written as ultradiscrete limit of determinant-type solutions of  $q$ -PII and the solutions whose initial value is perturbed. We also investigated some patterns of two-parameter solutions.

In the case of differential Painlevé equations, the variables of simultaneous equation are chosen to fit with the symplectic structure. We are not sure that

our choice of the variable  $y(q^{m+1}) = z(q^{m+1})z(q^m) + 1$  is a good  $q$ -deformation of the symplectic coordinate, and it would be desirable to find better choice of the variables.

One of the merit of studying ultradiscrete equations is that we may obtain exact solutions and it may help analysis of the original  $q$ -difference equations. To illustrate problems for future, we show another example of solution of p-ud PII with the parameter  $Q = -3$  and  $A = 7Q$ .

$$(\eta_m, Y_m), (\zeta_m, Z_m) = \quad (118)$$

$$\left\{ \begin{array}{lll} (+1, 6j - 35), & (+1, 3j - 38) & (m = -3j, j \geq 6) \\ (+1, 6j - 24), & (+1, 3j + 14) & (m = -3j + 1, j \geq 6) \\ (+1, 6j + 14), & (+1, 3j) & (m = -3j + 2, j \geq 6) \\ (+1, \underline{-5}), & (-1, -18) & (m = -15) \\ (-1, 11), & (+1, 29) & (m = -14) \\ (-1, 39), & (-1, 10) & (m = -13) \\ (+1, \underline{-11}), & (+1, -10) & (m = -12) \\ (+1, 16), & (+1, 26) & (m = -11) \\ (+1, 22), & (+1, -4) & (m = -10) \\ (+1, \underline{-17}), & (-1, 4) & (m = -9) \\ (-1, 27), & (+1, 23) & (m = -8) \\ (-1, \underline{-1}), & (-1, -23) & (m = -7) \\ (+1, \underline{-22}), & (+1, \underline{23}) & (m = -6) \\ (-1, 37), & (-1, 14) & (m = -5) \\ (+1, \underline{-29}), & (+1, -14) & (m = -4) \\ (+1, 6), & (+1, \underline{20}) & (m = -3) \\ (-1, \underline{-9}), & (-1, -20) & (m = -2) \\ (+1, \underline{-26}), & (+1, \underline{20}) & (m = -1) \\ (-1, 11), & (-1, -9) & (m = 0) \\ (+1, \underline{-41}), & (+1, \underline{9}) & (m = 1) \\ (-1, 14), & (-1, \underline{5}) & (m = 2) \\ (+1, \underline{-47}), & (+1, \underline{-5}) & (m = 3) \\ (-1, 8), & (-1, \underline{13}) & (m = 4) \\ (+1, \underline{-53}), & (+1, \underline{-13}) & (m = 5) \\ (-1, 2), & (-1, \underline{15}) & (m = 6) \\ (+1, \underline{-6j - 41}), & (+1, \underline{-15}) & (m = 2j + 1, j \geq 3) \\ (-1, \underline{-6j + 20}), & (-1, \underline{15}) & (m = 2j, j \geq 4) \end{array} \right.$$

If  $m \leq -16$  (resp.  $m \geq 7$ ), then the solution is written in the form of Proposition 11 (resp. Proposition 10). In the case  $-15 \leq m \leq -8$  (resp.  $1 \leq m \leq 7$ ), the solution is written in the form of Eqs.(76), (78) (resp. Eqs.(83), (84)). We underline the values such that the inequality  $Y_m < 0$  or  $mQ - Z_m < 0$  holds. It is apparent that the tendency that the inequality  $Y_m < 0$  or  $mQ - Z_m < 0$  holds increases as the value  $m$  increases, and it is also apparent for solutions in Eqs.(101), (104), (114), (117).

We conjecture that, if a solution of p-ud PII does not have the forward indefinite evolution and the backward indefinite evolution for all  $m \in \mathbb{Z}$ , then

there exist some values  $m'$  and  $m''$  such that the solution is written in the form of Proposition 11 for  $m < m'$  and in the form of Proposition 10 for  $m > m''$ . Then we propose a problem that how the asymptotics as  $m \rightarrow -\infty$  is concerned with the asymptotics as  $m \rightarrow +\infty$ . In the example given in Eq.(118), the asymptotics

$$(\eta_m, Y_m), (\zeta_m, Z_m) = \begin{cases} (+1, 6j - 35), & (+1, 3j - 38) & (m = -3j, j \geq 6) \\ (+1, 6j - 24), & (+1, 3j + 14) & (m = -3j + 1, j \geq 6) \\ (+1, 6j + 14), & (+1, 3j) & (m = -3j + 2, j \geq 6) \end{cases} \quad (119)$$

as  $m \rightarrow -\infty$  is connected with the asymptotics

$$(\eta_m, Y_m), (\zeta_m, Z_m) = \begin{cases} (-1, -6j + 20), & (-1, 15) & (m = 2j, j \geq 3) \\ (+1, -6j - 41), & (+1, -15) & (m = 2j + 1, j \geq 3) \end{cases} \quad (120)$$

as  $m \rightarrow +\infty$ .

In order to study solutions of  $q$ -Painlevé equations of other types, ultradiscretization with parity variables of them and analysis of the ultradiscrete solutions should be performed. Note that p-ud Painlevé III (p-ud Painlevé VI) was derived in [4] (resp. [10]).

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## A Summary of ultradiscrete limit of determinant-type solutions in [2].

It is shown in [1] that q-P II (Eq.(1)) with  $a = q^{2N+1}$  ( $N \in \mathbb{Z}$ ) admits a class of special solutions. Eq.(1) with  $a = q^{2N+1}$  is solved by

$$z^{(N)}(m) = \begin{cases} \frac{g^{(N)}(m)g^{(N+1)}(m+1)}{q^N g^{(N)}(m+1)g^{(N+1)}(m)} & (N \geq 0), \\ \frac{g^{(N)}(m)g^{(N+1)}(m+1)}{q^{N+1} g^{(N)}(m+1)g^{(N+1)}(m)} & (N < 0), \end{cases} \quad (121)$$



$$g^{(N)}(m) = \begin{cases} \begin{vmatrix} w(m) & w(m+2) & \cdots & w(m+2N-2) \\ w(m-1) & w(m+1) & \cdots & w(m+2N-3) \\ \vdots & \vdots & \ddots & \vdots \\ w(m-N+1) & w(m-N+3) & \cdots & w(m+N-1) \end{vmatrix} & (N > 0), \\ 1 & (N = 0), \\ \begin{vmatrix} w(m-1) & w(m-3) & \cdots & w(m-2|N|+1) \\ w(m) & w(m-2) & \cdots & w(m-2|N|+2) \\ \vdots & \vdots & \ddots & \vdots \\ w(m+|N|-2) & w(m+|N|-4) & \cdots & w(m-|N|) \end{vmatrix} & (N < 0) \end{cases} \quad (122)$$

where  $w(m)$  is a solution of  $q$ -Airy equation

$$w(m+1) - q^m w(m) + w(m-1) = 0. \quad (123)$$

We review the ultradiscrete limit of the determinant-type solutions  $z^{(M)}(m)$  for  $M \in \mathbb{Z}_{<0}$  by following section 4 of [2]. Recall that the  $q$ -Airy equation has special solutions, the  $q$ -Ai function  $a(m)$  and the  $q$ -Bi function  $b(m)$  (see [3] for the expression), and the general solution is given by the linear combination. We express a solution of the  $q$ -Airy equation as

$$w(m) = \alpha e^{A'/\varepsilon} a(m) + \beta e^{B'/\varepsilon} b(m), \quad (124)$$

where  $\alpha, \beta \in \{\pm 1\}$  and  $A', B' \in \mathbb{R}$ .

Let  $M \in \mathbb{Z}_{<0}$  and set  $N = -M$ . The  $p$ -ultradiscrete analogue of  $g^{(N)}(m)$  for  $N \in \mathbb{Z}_{>0}$  was calculated in [3, Proposition 3]. On the other hand, it follows from Eq.(122) that

$$g^{(M)}(m) = g^{(N)}(m - N). \quad (125)$$

By combining them, we have

**Proposition 12.** For  $m \leq M + 1$ , the  $p$ -ultradiscrete analogue of  $g^{(M)}(m)$  is given by

$$(\gamma^{(M)}(m), G^{(M)}(m)) = \begin{cases} (\gamma_0^{(M)}(m), G_0^{(M)}(m)) & (B' - A' < f_1^{(M)}(m)) \\ (\gamma_k^{(M)}(m), G_k^{(M)}(m)) & (f_k^{(M)}(m) \leq B' - A' < f_{k+1}^{(M)}(m)) \\ (\gamma_{-M}^{(M)}(m), G_{-M}^{(M)}(m)) & (f_{-M}^{(M)}(m) \leq B' - A'), \end{cases} \quad (126)$$

where

$$\gamma_k^{(M)}(m) = (-1)^{Mk-k(k+1)/2} \alpha^{M-k} \beta^k, \quad (127)$$

$$\begin{aligned} G_k^{(M)}(m) = & (-M-k)A' + kB' + \left[ \left( -k - \frac{1}{2}M \right) m^2 \right. \\ & + \left\{ -3k^2 + (-4M+1)k - \frac{M(M-2)}{2} \right\} m - \frac{8}{3}k^3 \\ & \left. + \left( -5M + \frac{3}{2} \right) k^2 + \left( -3M^2 + 2M + \frac{1}{6} \right) k - \frac{M(M-1)(M-2)}{6} \right] Q, \end{aligned} \quad (128)$$

$$\begin{aligned} f_k^{(M)}(m) = & G_{k-1}^{(M)}(m) - G_k^{(M)}(m) - A' + B' \\ = & \{ (m+2M+3k-2)^2 - (M+k)(M+k-1) \} Q, \end{aligned} \quad (129)$$

$k = 0, 1, \dots, -M$ .

We may write the p-ultradiscrete analogue of (121) as follows:

$$\begin{aligned} \zeta^{(M)}(m) = & \gamma^{(M+1)}(m+1) \gamma^{(M+1)}(m) \gamma^{(M)}(m+1) \gamma^{(m)}(m), \\ Z^{(M)}(m) = & G^{(M+1)}(m+1) - G^{(M+1)}(m) - G^{(M)}(m+1) \\ & + G^{(M)}(m) - (M+1)Q. \end{aligned} \quad (130)$$

For  $m \leq M+1$ , we set

$$\begin{aligned} h_{I,l}^{(M)}(m) = & f_l^{(M)}(m), \quad h_{II,l}^{(M)}(m) = f_l^{(M)}(m+1), \\ h_{III,l}^{(M)}(m) = & f_l^{(M+1)}(m), \quad h_{IV,l}^{(M)}(m) = f_l^{(M+1)}(m+1), \end{aligned} \quad (131)$$

where  $f_l^{(M)}(m)$  was defined in (129). Then we have

$$h_{I,l}^{(M)}(m) < h_{II,l}^{(M)}(m) < h_{III,l}^{(M)}(m) < h_{IV,l}^{(M)}(m) = h_{I,l+1}^{(M)}(m) \quad (132)$$

for  $l = 1, \dots, -M$ .

**Proposition 13.** For  $m \leq M+1$ , we have

$$\begin{aligned} (\zeta^{(M)}(m), Z^{(M)}(m)) = & \\ \begin{cases} (+1, (-m-2M-1)Q) & (B' - A' < h_{I,1}^{(M)}(m)) \\ (\zeta_{I,l}^{(M)}(m), Z_{I,l}^{(M)}(m)) & (h_{I,l}^{(M)}(m) \leq B' - A' < h_{II,l}^{(M)}(m)) \\ (\zeta_{II,l}^{(M)}(m), Z_{II,l}^{(M)}(m)) & (h_{II,l}^{(M)}(m) \leq B' - A' < h_{III,l}^{(M)}(m)) \\ (\zeta_{III,l}^{(M)}(m), Z_{III,l}^{(M)}(m)) & (h_{III,l}^{(M)}(m) \leq B' - A' < h_{I,l+1}^{(M)}(m)) \\ (+1, mQ) & (h_{II,-M}^{(M)}(m) \leq B' - A'), \end{cases} \end{aligned} \quad (133)$$

where

$$\begin{cases} \zeta_{\text{I},l}^{(M)}(m) = (-1)^{M+l}\alpha\beta, \\ Z_{\text{I},l}^{(M)}(m) = B' - A' - \{m^2 + (6l + 4M - 3)m \\ \quad + 8l^2 + (10M - 7)l + 3M^2 - 5M + 1\}Q, \end{cases} \quad (134)$$

$$\begin{cases} \zeta_{\text{II},l}^{(M)}(m) = +1, \\ Z_{\text{II},l}^{(M)}(m) = (m + 2M + 2l)Q, \end{cases} \quad (135)$$

$$\begin{cases} \zeta_{\text{III},l}^{(M)}(m) = (-1)^{M+l+1}\alpha\beta, \\ Z_{\text{III},l}^{(M)}(m) = A' - B' + \{m^2 + (6l + 4M + 1)m + 8l^2 + (10M + 1)l + 3M^2 + M\}Q, \end{cases} \quad (136)$$

for  $l = 1, 2, \dots, -M$ .

We assume that the values of  $A', B'$  and  $Q$  are chosen as  $m_0$  satisfies

$$m_0 \leq \min[3M + 1, -M(M + 1)/2 - 1], \quad m_0^2 Q \leq B' - A' < (m_0 + 1)^2 Q \quad (137)$$

Set

$$\begin{aligned} P_0 &:= (m_0 + 1)^2 Q, \\ P_j &:= \{m_0^2 - (M + j)(M + j - 1)\}Q \quad (j = 1, \dots, -M) \end{aligned} \quad (138)$$

Then, there exists an integer  $k_0 \in \{0, 1, \dots, -M - 1\}$  such that  $P_{k_0+1} \leq B' - A' < P_{k_0}$  holds. Using these notation, the result is written as follows:

**Theorem 3.** (c.f. [2, Theorem 5]) Assume that we have  $m_0$  and  $k_0$  mentioned above for assigned values of  $A', B'$  and  $Q$ . Then the following function  $(\zeta^{(M)}(m), Z^{(M)}(m))$  is obtained by the p-ultradiscrete limit of the solution of  $q$ -PII in terms of determinants.

(I) If  $m \leq m_0 + M$ , then

$$(\zeta^{(M)}(m), Z^{(M)}(m)) = (+1, mQ). \quad (139)$$

(II) If  $m_0 + M + 1 \leq m \leq m_0 - 2M - 3k_0$  ( $k_0 \neq 0$ ) or  $m_0 + M + 1 \leq m \leq m_0 - 2M - 1$  ( $k_0 = 0$ ), then

$$\begin{aligned} (\zeta^{(M)}(m), Z^{(M)}(m)) = & \\ \begin{cases} (+1, (m_0 + M + j)Q) & (m = m_0 + M + 3j - 2) \\ ((-1)^{j+1}\alpha\beta, B' - A' - (m_0^2 + m_0 - j^2 - M)Q) & (m = m_0 + M + 3j - 1) \\ ((-1)^{j+1}\alpha\beta, A' - B' + (m_0^2 + m_0 - j^2 + 2j + M)Q) & (m = m_0 + M + 3j), \end{cases} \end{aligned} \quad (140)$$

where  $1 \leq j \leq -M - k_0$ .

(III) If  $m_0 - 2M - 3k_0 + 1 \leq m \leq m_0 - 2M - 1$  and  $k_0 \neq 0$ , then

$$(\zeta^{(M)}(m), Z^{(M)}(m)) = \begin{cases} ((-1)^{j+1}\alpha\beta, B' - A' - (m_0^2 - m_0 - j^2 - M)Q) & (m = m_0 + M + 3j - 2) \\ ((-1)^{j+1}\alpha\beta, A' - B' + (m_0^2 - m_0 - j^2 + 2j + M)Q) & (m = m_0 + M + 3j - 1) \\ (+1, (m_0 + M + j)Q) & (m = m_0 + M + 3j), \end{cases} \quad (141)$$

where  $-M - k_0 + 1 \leq j \leq -M$  for the first case and  $-M - k_0 + 1 \leq j \leq -M - 1$  for the second and the third cases.

(IV) If  $m_0 - 2M - 1 \leq m \leq M + 1$  ( $k_0 \neq 0$ ) or  $m_0 - 2M \leq m \leq M + 1$  ( $k_0 = 0$ ), then

$$(\zeta^{(M)}(m), Z^{(M)}(m)) = (+1, (-m - 2M - 1)Q). \quad (142)$$

By setting  $C = B' - A' - (m_0^2 + m_0)Q$ , we obtain Proposition 7.

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